SMALL SURREAL FIELDS CLOSED UNDER EXPONENTIAL AND LOGARITHM

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ABSTRACT. The class of surreal numbers, denoted by **No**, initially considered by Conway, can be seen as a universal ordered field: if we forget that it is a class but not a set, it is a field, and any ordered field can be embedded into it. In particular, surreal numbers include all real numbers but also ordinal numbers. Surreal numbers are known to have strong relations with other mathematical objects, such as the field of transseries.

In this article, we exhibit a class of surreal fields that are closed under both exponential and logarithmic functions. We actually present a strict hierarchy of such fields inside the already known fields of literature.

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1. Introduction

Conway introduced the class of surreal numbers in [6]. The class of surreal numbers were later on popularised by Knuth [14], and then formalised later on by Gonshor in [11], and then studied by many other authors. The initial idea is to define a class of numbers based on a suitable concept of "simplicity". This leads to obtaining a real closed field (if we allow fields to be built on a class and not a set) that contains both the real numbers and the ordinals: indeed, the idea of Conway provides a way to unify Dedekind's construction of real numbers in terms of cuts of the rational numbers and von Neumann's construction of ordinal numbers by transfinite induction in terms of set membership.

As stated by [8], the class of surreal numbers can be considered to be "the" field that includes "all numbers great and small". In particular, any divisible ordered Abelian group is isomorphic to an initial subgroup of **No**, and any real closed field is isomorphic to an initial subfield of **No** [20, Theorems 9 and 19], [6, Theorems 28 and 29].

Following the alternative presentation from Gonshor in [11], based on some ideas from Conway [6], a surreal number can also be seen as an ordinal-length sequence over $\{+,-\}$, that we call a **sign sequence**. We call the ordinal length of the sign sequence the **length of the surreal number**: we write $|x|_{+-}$ for the length of the surreal number x. Basically, the idea is that such sequences are ordered lexicographically and have a tree-like structure. Namely, a $+$ (respectively $-$) added to a sequence x denotes the simplest number greater (resp. smaller) than x but smaller (resp. greater) than all the prefixes of x which are greater (resp. smaller) than x. With this definition of surreal numbers, it is possible to retrieve Conway's original notion of cuts and the corresponding notion of simplicity. Using cuts, it is also possible to define operations such as addition, subtraction, multiplication, and division, providing a real closed field.

This vision of the surreal number leads naturally to consider some substructures by fixing a bound on the length of the sign sequence. In that spirit, we write $\mathbf{No}_\lambda$ for the set of the surreal numbers whose length is less than ordinal λ. The algebraic structure of $\mathbf{No}_\lambda$ is well known: it is a group, a ring or a field *iff* λ is an additive[1] ordinal, a multiplicative[2] ordinal or an ε-number[3] respectively ([20, 19, Corollaries 3.1 and 3.4, Proposition 4.7]).

Following Gonshor [11], based on ideas from Kruskal, it is also possible to define in a natural way some functions, such as the exponential function and the logarithmic function over **No**. The obtained functions are extensions of the classical corresponding functions over the real numbers and provide a way to extend some statements from classical mathematical analysis to this field of numbers.

The class **No** can also be equipped with a derivation ∂ so that it can be considered as a field of transseries [3, 4]. See [18] for a survey of recent results in all these directions. The class **No** can also be seen as a field of (generalised) power series with real coefficients, namely as Hahn series, where exponents are surreal numbers

[1]An ordinal α is **additive** if for all $\beta, \gamma < \alpha$, $\beta + \gamma < \alpha$. It can we shown that it is equivalent to the fact that $\alpha = \omega^\beta$ for some ordinal β.

[2]An ordinal α is **multiplicative** if for all $\beta, \gamma < \alpha$, $\beta\gamma < \alpha$. It can we shown that it is equivalent to the fact that $\alpha = \omega^{\omega^\beta}$ for some ordinal β.

[3]An ordinal α is an ε**-number** if it satisfies $\alpha = \omega^\alpha$. In particular it is both additive and multiplicative.

themselves. More precisely, write $\mathbb{K}((G))$ for the set (resp. class) of Hahn series with coefficients in $\mathbb{K}$ and terms corresponding to elements of G, where $\mathbb{K}$ is a field, and G is some divisible ordered Abelian group (resp. class-group): This means that $\mathbb{K}((G))$ corresponds to formal power series of the form $s = \sum_{g\in S} a_g t^g$, where S is a well-ordered subset of G and $a_g \in \mathbb{K}$. The support of s is $\text{supp}(s) = \{g \in S \mid a_g \neq 0\}$ and the length of the serie of s is the order type of supp(s). Note that in this definition, even if G is a class, we require S to be a set. The field operations on $\mathbb{K}((G))$ are defined as expected, considering elements of $\mathbb{K}((G))$ as formal power series. With this formalism, it appears that, up to isomorphism, the following holds: $\mathbf{No} = \mathbb{R}((\mathbf{No}))$. Indeed, as proven by Conway [6] and Gonshor [11], every surreal number has a normal form: it can be uniquely written as $x = \sum\limits_{i<\nu} r_i\omega^{a_i}$ where ν is an ordinal, $(r_i)_{i<\nu}$ are non-zero real numbers and $(a_i)_{i<\nu}$ is a decreasing sequence of surreal numbers. We will call ν the **series-length** of x and will denote $\nu(x)$.

In this article, we show that there are "small" acceptable subfields of **No** that are closed under exponential and logarithmic functions. These fields are built with some restrictions on ordinals allowed to get involved in either the length of the Hahn series or in its exponents.

Given some ordinal γ (or more generally a class of ordinals), we write $\mathbb{K}((G))_\gamma$ for the restriction of $\mathbb{K}((G))$ to formal power series whose support has an order type in γ (that is to say, corresponds to some ordinal less than γ). We have of course $\mathbf{No} = \mathbb{R}((\mathbf{No}))_{\mathbf{Ord}}$. In this point of view, **ε-numbers**, *i.e* ordinals λ, such that $\omega^\lambda = \lambda$, play a major role as they are essential to handle field operations and the exponential function.

As we will often play with exponents of formal power series considered in the Hahn series, we introduce the following notation: We denote

$$\mathbb{R}^\Gamma_\lambda = \mathbb{R}((\Gamma))_\lambda$$

when λ is an ε-number and Γ a divisible Abelian group.

From MacLane's theorem ([17, Theorem 1], see also [1, Section 6.23]), we know that $\mathbb{R}^{\mathbf{No}_\mu}_\lambda$ is a real-closed field when μ is a **multiplicative ordinal** (i.e. $\mu = \omega^{\omega^\alpha}$ for some ordinal α) and λ an ε-number.

We now introduce some notations to state our main theorems.

We first need to associate a canonical sequence to any ε-number.

Definition 1.1 (Canonical sequence defining an ε-number)**.** Let λ be an ε-number. Ordinal λ can always be written as $\lambda = \sup\,(e_\beta)_{\beta<\gamma_\lambda}$ for some **canonical sequence**, where γ_λ is the length of this sequence, and this sequence is defined as follows:

- If $\lambda = \varepsilon_0$ then we can write $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$ and we take $\omega, \omega^\omega, \omega^{\omega^\omega}, \dots$ as canonical sequence for ε_0. Its length is ω, and for $\beta < \lambda$, e_β is $\omega^{\cdot^{\cdot^{\cdot^\omega}}}$ where there are β occurrences of ω in the exponent.
- If $\lambda = \varepsilon_\alpha$, where α is a non-zero limit ordinal, then we can write $\lambda = \sup\limits_{\beta<\alpha}\varepsilon_\beta$ and we take $(\varepsilon_\beta)_{\beta<\alpha}$ as the canonical sequence of λ. Its length is α and for $\beta < \alpha$, $e_\beta = \varepsilon_\beta$.
- If $\lambda = \varepsilon_\alpha$, where α is a successor ordinal, then we can write

$$\lambda = \sup\{\varepsilon_{\alpha-1}, {\varepsilon_{\alpha-1}}^{\varepsilon_{\alpha-1}}, {\varepsilon_{\alpha-1}}^{{\varepsilon_{\alpha-1}}^{\varepsilon_{\alpha-1}}}, \dots\}$$

and we take $\varepsilon_{\alpha-1}, {\varepsilon_{\alpha-1}}^{\varepsilon_{\alpha-1}}, {\varepsilon_{\alpha-1}}^{{\varepsilon_{\alpha-1}}^{\varepsilon_{\alpha-1}}}, \dots$ as the canonical sequence of λ. Its length is ω, and for $\beta < \omega$, $e_\beta = {\varepsilon_{\alpha-1}}^{\cdot^{\cdot^{\cdot^{\varepsilon_{\alpha-1}}}}}$ where there are β occurrences of $\varepsilon_{\alpha-1}$ in the exponent.

For example, here are some canonical sequences for some ε-numbers:

ε-number	Canonical sequence
ε_0	$\omega, \omega^\omega, \omega^{\omega^\omega}, \dots$
ε_1	$\varepsilon_0, {\varepsilon_0}^{\varepsilon_0}, {\varepsilon_0}^{{\varepsilon_0}^{\varepsilon_0}}, \dots$
ε_2	$\varepsilon_1, {\varepsilon_1}^{\varepsilon_1}, {\varepsilon_1}^{{\varepsilon_1}^{\varepsilon_1}}, \dots$
$\vdots$	$\vdots$
ε_ω	$\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$
$\vdots$	$\vdots$
$\varepsilon_{\omega 2}$	$\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_\omega, \varepsilon_{\omega+1}, \dots$
$\varepsilon_{\omega 2+1}$	$\varepsilon_{\omega 2}, {\varepsilon_{\omega 2}}^{\varepsilon_{\omega 2}}, {\varepsilon_{\omega 2}}^{{\varepsilon_{\omega 2}}^{\varepsilon_{\omega 2}}}, \dots$
$\vdots$	$\vdots$

Main contributions of this paper.

Definition 1.2. Let Γ be an Abelian subgroup of $\mathbf{No}$ and λ be an ε-number whose canonical sequence is $(e_\beta)_{\beta<\gamma_\lambda}$. We denote $\Gamma^{\uparrow\lambda}$ for the family of group $(\Gamma_\beta)_{\beta<\gamma_\lambda}$ defined as follows:

- $\Gamma_0 = \Gamma$;
- $\Gamma_{\beta+1}$ is the group generated by Γ_β, $\mathbb{R}_{e_\beta}^{g\left((\Gamma_\beta)_+^*\right)}$ and $\left\{ h(a_i) \;\middle|\; \sum_{i<\nu} r_i \omega^{a_i} \in \Gamma_\beta \right\}$ where g and h are Gonshor's functions associated to exponential and logarithm (see Section 3.2 below or [11] for some details);
- For a limit ordinal number β, $\Gamma_\beta = \bigcup_{\gamma<\beta} \Gamma_\gamma$.

When considering a family of set $(S_i)_{i\in I}$, we denote

$$\mathbb{R}_\lambda^{(S_i)_{i\in I}} = \bigcup_{i\in I} \mathbb{R}_\lambda^{S_i}$$

in particular,

$$\mathbb{R}_\lambda^{\Gamma^{\uparrow\lambda}} = \bigcup_{i<\gamma_\lambda} \mathbb{R}_\lambda^{\Gamma_i}$$

Remark. By construction, if $\Gamma \subseteq \Gamma'$ then $\mathbb{R}_\lambda^{\Gamma^{\uparrow\lambda}} \subseteq \mathbb{R}_\lambda^{\Gamma'^{\uparrow\lambda}}$.

Remark. The idea behind the definition of $\Gamma^\uparrow$ is that at step $i+1$ we add new elements that we will prove to yield closure of $\mathbb{R}_\lambda^{\Gamma_i}$ under exponential and logarithm. The reason why we add $\mathbb{R}_{e_\beta}^{g\left((\Gamma_\beta)_+^*\right)}$ to Γ_β rather than $\mathbb{R}_\lambda^{g\left((\Gamma_\beta)_+^*\right)}$ is that we want to keep control on what we add in the new group.

With these notations, we are now ready to state our main theorems. The three following theorems decompose $\mathbf{No}_\lambda$ into a strictly increasing hierarchy of subfields, each of them being closed under exponential and logarithm.

Theorem 1.3. *Let Γ be an Abelian subgroup of* **No** *and λ be an ε-number, then $\mathbb{R}_\lambda^{\Gamma^{\uparrow\lambda}}$ is closed under exponential and logarithmic functions.*

Theorem 1.4. *Let λ be an ε-number. $\mathbf{No}_\lambda = \bigcup_\mu \mathbb{R}_\lambda^{{\mathbf{No}_\mu}^{\uparrow\lambda}}$, where μ ranges over the additive ordinals less than λ (equivalently, μ ranges over the multiplicative ordinals less than λ).*

Theorem 1.5. *For all ε-number λ, the hierarchy in Theorem 1.4 is strict:*

$$\mathbb{R}_\lambda^{{\mathbf{No}_\mu}^{\uparrow\lambda}} \subsetneq \mathbb{R}_\lambda^{{\mathbf{No}_{\mu'}}^{\uparrow\lambda}}$$

for all multiplicative ordinals μ and μ' such that $\omega < \mu < \mu' < \lambda$.

Relation to state of the art. This hierarchy is to be compared with the one given by [20, Proposition 4.7], which states

$$\mathbf{No}_\lambda = \bigcup_\mu \mathbb{R}_\lambda^{\mathbf{No}_\mu}$$

where again μ ranges over the additive ordinals less than λ (equivalently, μ ranges over the multiplicative ordinals less than λ). Although $\mathbf{No}_\lambda$ is indeed closed under exponential and logarithmic functions ([19, Corollary 5.5]), none of the fields $\mathbb{R}_\lambda^{\mathbf{No}_\mu}$ is:

Proposition 1.6. *$\mathbb{R}_\lambda^{\mathbf{No}_\mu}$ is never closed under exponential function for $\mu < \lambda$ a multiplicative ordinal.*

Under the assumption of Theorem 1.3, the fields $\mathbb{R}_\lambda^{\Gamma^{\uparrow\lambda}}$ are closed under exp and ln.

Organization of the paper. The remaining of the paper is organised as follows:

- In Section 2, we introduce the notations we use about well-ordered sets and ordinals numbers. We also state some known useful results about them.
- In Section 3, we present all the properties about surreal numbers. It contains mostly already known results but also some new useful lemmas such as Lemmas and 3.23 and 3.24 with their Corollary 3.25.
- Section 4 is dedicated to the proof of Theorems 1.3, 1.4 and 1.5 and also Proposition 1.6.

2. Well ordered sets toolbox

Before working with surreal numbers, we must state some statements about ordinal numbers and well-ordered sets. We assume some familiarity with these concepts. For an introduction, see for instance [5, 7, 10, 12, 16].
In this section, we introduce the notations we use for operations over ordinal numbers, and we state a useful technical proposition we will use later.
Considering ordered sets, we need the following notions.

Definition 2.1. Let $(S, \leq)$ a (partially) ordered set and $A, B \subseteq S$. A and B are **cofinal** if for any $b \in B$, there is $a \in A$ such that $b \leq a$ and for any $a \in A$ there is $b \in B$ such that $a \leq b$. Similarly, A and B are **coinitial** if for any $b \in B$ there is $a \in A$ such that $a \leq b$ and for any $a \in A$ there is $b \in B$ such that $b \leq A$.

Example. For example, $\{0\}$ and $\mathbb{N}$ are coinitial. Also, $\mathbb{Z}$ and $\mathbb{R}$ are coinitial and cofinal.

Well-orders and ordinal numbers are strongly related. Through the following famous statement, ordinal numbers might be considered as canonical representatives of well-ordered sets (see, e.g. [13] for proofs of the statements of this section unless an explicit reference to a proof in literature is mentioned).

Theorem 2.2. *Let $(X, <)$ be a well total order. There is a unique ordinal which is order-isomorphic to $(X, <)$.*

Definition 2.3 (Order type)**.** We call the ordinal which isomorphic to a well ordered set, its **order type**.

Cardinal numbers can be seen as special ordinal numbers.

Definition 2.4 (Cardinal)**.** A **cardinal** κ is an ordinal such that for all $\alpha \in \kappa$, there is no bijection between α and κ. A cardinal is **regular** if for any morphism φ between $\alpha \in \kappa$ and κ, φ is bounded above, *i.e* there is some $\beta \in \kappa$ such that for all $\gamma \in \alpha$, $\varphi(\gamma) \in \beta$.

Example. ω is regular, as well as ω_1 the first uncountable ordinal.

Cardinal ω_1 and other regular cardinals can be considered as "huge" ordinals. In fact, they can be associated with models of the Set Theory [13]. The literature has considered regular cardinals in order to define subfields of surreal numbers with exponential and logarithm [15], or for generalisations of $\mathbb{R}$ [9]. In this article, our aim is to define some subfields with exponential and logarithm functions but with "small" ordinals.
We will consider ordinals with their classical operations, such as addition, multiplication, and exponentiation.

Definition 2.5 (Usual operations)**.** The usual addition, $\oplus$, the usual multiplication $\otimes$ and the exponentiation over ordinal numbers are defined as follows: for all ordinal numbers α and β,

- $\alpha \oplus 0 = \alpha$.
- $\alpha \oplus 1 = \alpha \cup \{\alpha\}$.
- $\alpha \oplus (\beta \oplus 1) = (\alpha \oplus \beta) \oplus 1$
- If β is a limit ordinal, $\alpha \oplus \beta = \sup \{ \alpha \oplus \gamma \mid \gamma < \beta \}$.
- $\alpha \otimes 0 = 0$
- $\alpha \otimes (\beta \oplus 1) = (\alpha \otimes \beta) \oplus \alpha$
- If β is a limit ordinal, $\alpha \otimes \beta = \sup \{ \alpha \otimes \gamma \mid \gamma < \beta \}$.
- $\alpha^0 = 1$
- $\alpha^{\gamma \oplus 1} = \alpha^\gamma \otimes \alpha$
- If β is a limit ordinal, $\alpha^\beta = \sup \{ \alpha^\gamma \mid \gamma < \beta \}$.

These operations are not commutative and $\otimes$ is not distributive over $\oplus$.
We will also need to talk about the so-called natural (or Hessenberg) addition and multiplications, which is a nice generalisation of the corresponding operations over integers based on Cantor's Normal Form. Formally.

Theorem 2.6 (Cantor, [5])**.** *Let $\beta \geq 2$ and α be ordinal numbers. Then α can be written in a unique way as*

$$\beta^{e_1} k_1 \oplus \beta^{e_2} k_2 \oplus \cdots \oplus \beta^{e_n} k_n$$

with $0 \leq k_i < \beta$ and $e_1 > e_2 > \cdots > e_n > 0$ being ordinal numbers.

We call such an expression the **normal form in base** β of the ordinal α. The special case $\beta = \omega$ is called the **Cantor's normal form**.

Definition 2.7 (Natural[4] addition and multiplication, [12]). Let $\alpha = \sum_{i=1}^{n} \omega^{\alpha_i} k_i$ and $\beta = \sum_{j=1}^{m} \omega^{\beta_j} l_j$ be two ordinal numbers in Cantor's normal form. Let p the number of element in the finite set $\{\alpha_i\}_i \cup \{\beta_j\}_j$. Let $\gamma_1 > \cdots > \gamma_p$ a decreasing enumeration of this set. Up to set $k_i = 0$ or $l_j = 0$, we assume, without loss of generality $n = m = p$ and for all $i \leq p$ $\alpha_i = \beta_i = \gamma_i$. We then define natural addition by

$$\alpha + \beta = \sum_{i=1}^{p} \omega^{\gamma_i}(k_i + l_i)$$

and natural multiplication by

$$\alpha \times \beta = \sum_{i,j} \omega^{\alpha_i + \beta_j} k_i l_j$$

where the symbol $\sum$ denotes a natural addition.

We finish this section with a useful proposition about the order type of a monoid.

Proposition 2.8 ([21, Weiermann, Corollary 1]). *Let Γ be an ordered Abelian group and $S \subseteq \Gamma_+$ be a well-ordered subset with order type α. Then, $\langle S \rangle$, the monoid generated by S in Γ is itself well-ordered with order type at most $\omega^{\widehat{\alpha}}$ where, if the Cantor normal form of α is*

$$\alpha = \sum_{i=1}^{n} \omega^{\alpha_i} n_i$$

then

$$\widehat{\alpha} = \sum_{i=1}^{n} \omega^{\alpha'_i} n_i$$

and

$$\beta' = \begin{cases} \beta + 1 & \text{if } \beta \text{ is an } \varepsilon\text{-number} \\ \beta & \text{otherwise} \end{cases}$$

In particular, $\langle S \rangle$ has order type at most $\omega^{\omega\alpha}$ (commutative multiplication).

3. Surreal numbers toolbox

3.1. **Generalities.** Surreal numbers were first introduced by Conway [6] using an approach based on cuts, unifying the approach of Dekekind's cuts for defining real numbers, and Von Neuman's approach for defining ordinal numbers. Conway also suggested a representation using a sign sequence. The approach based on sign sequences was later fully formalised and more deeply explored by Gonshor. In particular, this is the main approach for all definitions and all fundamental constructions in [11].

Following this view, a surreal number x can be seen as a function $x : \alpha \to \{+, -\}$ where α is an ordinal number. We call ordinal α the **length** of x, and we write it by $|x|_{+-}$. A surreal number y is a **prefix** of x if $|y|_{+-} \leq |x|_{+-}$ and

$$\forall \beta < |y|_{+-} \qquad y(\beta) = x(\beta)$$

In this case, we write $y \sqsubseteq x$. If moreover $y \neq x$ we write $y \sqsubset x$.

The alternative construction is based on cuts, and sureal numbers are seen as being born at some ordinal steps. A first surreal number of length 0 is born, then 1, etc.

[4] Also called Hessenberg addition and Hessenberg multiplication

After a surreal number of length ω appears and so on. More generally if L and R are two sets of previously defined surreal numbers such that $L < R$, *i.e*

$$\forall \ell \in L \quad \forall r \in R \qquad \ell < r$$

then $[L \mid R]$ is defined to be the shortest surreal number which lies in between L and R for the lexicographic order (with $- < \square < +$ and $\square$ is a blank symbol). In particular, $[\varnothing \mid \varnothing] = 0$ and 0 is identified with the empty sign sequence. If $x = [L \mid R]$ and $L \cup R = \{y \mid y \sqsubset x\}$ we say that $[L \mid R]$ is **the canonical representation** of x. Operations can be defined using these cuts as follows: for all surreal numbers $x, y \in \mathbf{No}$

$$x + y = \left[L^+_{x,y} \;\middle|\; R^+_{x,y}\right]$$

with
$$L^+_{x,y} = \left\{ x' + y, x + y' \;\middle|\; \begin{array}{cc} x' \sqsubset x & x' < x \\ y' \sqsubset y & y' < y \end{array} \right\}$$

and
$$R^+_{x,y} = \left\{ x'' + y, x + y'' \;\middle|\; \begin{array}{cc} x'' \sqsubset x & x < x'' \\ y'' \sqsubset y & y < y'' \end{array} \right\}$$

also
$$xy = \left[L^\times_{x,y} \;\middle|\; R^\times_{x,y}\right]$$

with
$$L^\times_{x,y} = \left\{ \begin{array}{c} x'y + xy' - x'y' \\ x''y + xy'' - x''y'' \end{array} \;\middle|\; \begin{array}{c} x', x'' \sqsubset x \\ x' < x < x'' \\ y', y' \sqsubset y \\ y' < y < y'' \end{array} \right\}$$

and
$$R^\times_{x,y} = \left\{ \begin{array}{c} x'y + xy'' - x'y'' \\ x''y + xy' - x''y' \end{array} \;\middle|\; \begin{array}{c} x', x'' \sqsubset x \\ x' < x < x'' \\ y', y'' \sqsubset y \\ y' < y < y'' \end{array} \right\}$$

Although it is not trivial, it can be checked that this defines field's[5] operations (see [11] for details).

With these operations, we can identify the integer n with the surreal number consisting of exactly n pluses. We use the same notation, n, to denote the corresponding surreal number. In fact, to be slightly more general, we can even identify all the surreal numbers whose lengths are finite to dyadic[6] numbers.

Having the integers we can now get a notion of order of magnitude. For instance, the surreal consisting in ω pluses is greater than any integer and thus has a higher order of magnitude.

Definition 3.1. We define the following relations for a and b two surreal numbers:

- $a \asymp b$ *iff* there is some natural number n such that $n|a| \geq |b|$ and $n|b| \geq |a|$. We say that a and b have the **same order of magnitude**.
- $a \prec b$ *iff* for all natural number n, $n|a| < |b|$. We say that b has a **higher order of magnitude** than a.
- $a \preceq b$ *iff* $a \prec b$ or $a \asymp b$. We say that b has **at least the same order of magnitude** as a.
- $a \sim b$ *iff* $a - b \prec 1$. We say that a and b are **equivalent**.

The relation $\preceq$ is a preorder and $\asymp$ and $\prec$ are the associated equivalence relation and strict preorder respectively. We can then try to look at the equivalence classes. The equivalence classes for $\asymp$ are called the **Archimedean classes**.

[5]Note that it is not a proper field since **No** is a proper class and not a set.

[6]Numbers of the form $k/2^n$ with $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Theorem 3.2 ([11, Gonshor, Theorem 5.1])**.** *For any non-zero surreal number a, there is a unique shortest positive element x such that $x \asymp a$. More precisely, if $y \asymp a$ and $y > 0$ then $x \sqsubseteq y$.*

Having such a candidate for a canonic representation of each equivalence class, we can ask for a better characterization. It turns out that it can be defined in an exponentiation-like way.

Definition 3.3. We define for all $x = [x' \mid x'']$ in canonical representation,

$$\omega^x = \left[0, \mathbb{R}_+^* \omega^{x'} \;\middle|\; \mathbb{R}_+^* \omega^{x''}\right]$$

An element of the form ω^x will be called a **monomial**.

Almost by definition, if $x < y$ then we have $\omega^x \prec \omega^y$ and it turns out that each ω^a represents its class for $\asymp$.

Theorem 3.4 ([11, Gonshor, Theorem 5.3] and [6, Conway, Theorem 19])**.** *A surreal number x is of the form $x = \omega^a$ if and only if it is the shortest positive element in its Archimedean class.*

Definition 3.5 ([11, Gonshor, page 59])**.** Let $(a_i)_{i<\nu}$ be an ordinal-length decreasing sequence of surreal numbers and $(r_i)_{i<\nu}$ be non-zero real numbers. We define by transfinite induction :

- $\sum\limits_{i<0} r_i \omega^{a_i} = 0$
- If $\nu = \nu' + 1$ then $\sum\limits_{i<\nu} r_i \omega^{a_i} = \sum\limits_{i<\nu'} r_i \omega^{a_i} + r_{\nu'} \omega^{a_{\nu'}}$
- If ν is a limit ordinal then

$$\sum_{i<\nu} r_i \omega^{a_i} = \left[\left\{ \sum_{i<\nu'} r_i \omega^{a_i} + (r_{\nu'} - \varepsilon) \omega^{a_{\nu'}} \;\middle|\; \begin{matrix} \nu' < \nu \\ \varepsilon \in \mathbb{R}_+^* \end{matrix} \right\} \middle| \left\{ \sum_{i<\nu'} r_i \omega^{a_i} + (r_{\nu'} + \varepsilon) \omega^{a_{\nu'}} \;\middle|\; \begin{matrix} \nu' < \nu \\ \varepsilon \in \mathbb{R}_+^* \end{matrix} \right\}\right]$$

If $x = \sum\limits_{i<\nu} r_i \omega^{a_i}$, we will call this writing the **Conway normal form** or simply the **normal form** of x. An element of the form $r\omega^a$ with $a \in \mathbf{No}$ and $r \in \mathbb{R}$ will be called a **term**. In particular, a monomial is a term. Finally, if $x = \sum\limits_{i<\nu} r_i \omega^{a_i}$ we denote $\nu(x) = \nu$ the length of the series in the normal form of x.

Theorem 3.6 ([11, Gonshor, Theorem 5.6] and [6, Conway, Theorem 21])**.** *Every surreal number can be uniquely expressed in the way $x = \sum\limits_{i<\nu} r_i \omega^{a_i}$.*

Definition 3.7 (Term)**.** Given a surreal number $a = \sum\limits_{i<\nu} r_i \omega^{a_i}$, for all $i < \nu$ we say that $r_i \omega^{a_i}$ is a **term** of a.

Definition 3.8. A surreal number a in normal form $a = \sum\limits_{i<\nu} r_i \omega^{a_i}$ is

- **purely infinite** if for all $i < \nu$, $a_i > 0$. If $\mathbb{K} \subseteq \mathbf{No}$ is a subfield of $\mathbf{No}$, we denote $\mathbb{K}_\infty$ the set (or class) of purely infinite numbers in $\mathbb{K}$. We also denote $\mathbb{K}_\infty^+$ the set (or class) of non-negative purely infinite numbers.
- **infinitesimal** if for all $i < \nu$, $a_i < 0$ (or equivalently if $a \prec 1$).
- **appreciable** if for all $i < \nu$, $a_i \leq 0$ (or equivalently if $a \preceq 1$).

If $\nu' \leq \nu$ is the first ordinal such that $a_i \leq 0$, then $\sum_{i<\nu'} r_i\omega^{a_i}$ is called the **purely infinite part** of a. Similarly, if $\nu' \leq \nu$ is the first ordinal such that $a_i < 0$, $\sum_{\nu' \leq i<\nu} r_i\omega^{a_i}$ is called the **infinitesimal part** of a.

In his book, Gonshor explains how to retrieve the sign sequence from the normal form. In the following, $a[:\alpha]$ is the prefix of length α of the surreal number a. Also, $|a|_+$ is the (ordinal) number of pluses in the sign sequence of a.

Theorem 3.9 ([11, Gonshor, Theorems 5.11 and 5.12])**.** *For a surreal number a,*

- *The signs sequence of ω^a is as follows: we start with a plus and for any ordinal $\alpha < |a|$ we add $\omega^{|a[:\alpha]|_+ +1}$ occurrences of $a(\alpha)$.*
- *The signs sequence of $\omega^a n$ is the signs sequence of ω^a followed by $\omega^{|a|_+}(n-1)$ pluses.*
- *The signs sequence of $\omega^a \dfrac{1}{2^n}$ is the signs sequence of ω^a followed by $\omega^{|a|_+}n$ minuses.*
- *The signs sequence of $\omega^a r$ for r a positive real is the signs sequence of ω^a to which we add each sign of r $\omega^{|a|_+}$ times excepted the first plus which is omitted.*
- *The signs sequence of $\omega^a r$ for r a negative real is the signs sequence of $\omega^a(-r)$ in which we change every plus in a minus and conversely.*
- *The signs sequence of $\sum_{i<\nu} r_i\omega^{a_i}$ is the juxtaposition of the signs sequences of the $\omega^{a_i^\circ} r_i$*

Lemma 3.10 ([20, van den Dries and Ehrlich, Lemma 4.1])**.** *For all surreal number $a \in \mathbf{No}$,*

$$|a|_{+-} \leq |\omega^a|_{+-} \leq \omega^{|a|_{+-}}$$

Lemma 3.11 ([11, Gonshor, Lemma 6.3] and [20, van den Dries and Ehrlich, Lemma 4.2])**.** *Let $x = \sum_{i<\nu} r_i\omega^{a_i}$ a surreal number. We have:*

- $\nu \leq |x|_{+-}$
- *for all $i < \nu$, $|r_i\omega^{a_i}|_{+-} \leq |x|_{+-}$*
- *if there is some α such that or all $i < \nu$, $|r_i\omega^{a_i}|_{+-} \leq \alpha$, then $|x|_{+-} \leq \alpha\nu$.*

3.2. **Exponential and logarithmic functions.** In his book [11], Gonshor shows that there are two functions $\exp : \mathbf{No} \to \mathbf{No}_+^*$ and $\ln : \mathbf{No}_+^* \to \mathbf{No}$ that are reciprocal to each other and that extend the corresponding functions over $\mathbb{R}$ with the usual properties about them. He also gave the definition of two useful auxiliary functions g and h that help to characterise exp and ln respectively. For a complete introduction of these functions see [11].

Definition 3.12 (Gonshor's g and h functions)**.** We fix the notation for the following functions:

- $g : \mathbf{No}_+^* \to \mathbf{No}$ satisfies for all $x \in \mathbf{No}_+^*$,

  $$g(x) = [\,c(x), g(x') \mid g(x'')]$$

  where $c(x)$ is the unique number such that $\omega^{c(x)} \asymp x$ and where x' ranges over the lower non-zero prefixes of x and x'' over the upper prefixes of x.

- $h : \mathbf{No} \to \mathbf{No}^*_+$ satisfies for all $x \in \mathbf{No}^*_+$
$$h(x) = \left[0, h(x') \;\middle|\; h(x''), \frac{\omega^b}{n}\right]$$
where x' ranges over the lower prefixes of x and x'' over the upper prefixes of x.

The exponential function then satisfies the following:

Theorem 3.13 ([11, Theorems 10.2, 10.3 and 10.4])**.** *For all $r \in \mathbb{R}$ and ε infinitesimal, we have*
$$\exp r = \sum_{k=0}^{\infty} \frac{r^k}{k!}, \quad \exp \varepsilon = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \quad \textit{and} \quad \exp(r+\varepsilon) = \exp(r)\exp(\varepsilon) = \sum_{k=0}^{\infty} \frac{(r+\varepsilon)^k}{k!}$$
Moreover for all purely infinite numbers x,
$$\exp(x + r + \varepsilon) = \exp(x)\exp(r+\varepsilon)$$

Proposition 3.14 (Function g, [11, Theorem 10.13])**.** *If $x = \sum_{i<\nu} r_i \omega^{a_i}$ is purely infinite, then*
$$\exp x = \omega^{\sum_{i<\nu} r_i \omega^{g(a_i)}}$$

The logarithmic function satisfies the following:

Proposition 3.15 ([11, Theorems 10.8, 10.9, 10.12 and 10.13])**.** *For all surreal number $a = \sum_{i<\nu} r_i \omega^{a_i}$, we have*
$$\ln \omega^a = \sum_{i<\nu} r_i \omega^{h(a_i)}$$

As a consequence of Theorem 3.13, ln satisfies the following

Corollary 3.16. *For x an infinitesimal,*
$$\ln(1+x) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1} x^i}{i}$$

Corollary 3.17. *Let $a = \sum_{i<\nu} r_i \omega^{a_i}$ be a positive surreal number. Then*
$$\ln a = \ln \omega^{a_0} + \ln r_0 + \ln\left(1 + \sum_{1 \le i < \nu} \frac{r_i}{r_0} \omega^{a_i - a_0}\right)$$

We will use the following properties of the functions g and h:

Proposition 3.18 ([11, Theorem 10.14])**.** *If a is an ordinal number then*
$$g(a) = \begin{cases} a+1 & \text{if } \lambda \le a < \lambda + \omega \text{ for some } \varepsilon\text{-number } \lambda \\ a & \text{otherwise} \end{cases}$$

Proposition 3.19 ([11, Theorem 10.15])**.** *Let n be a natural number and b be an ordinal. We have $g(2^{-n}\omega^{-b}) = -b + 2^{-n}$.*

Corollary 3.20. *If a is an ordinal number then $h(-a) = \omega^{-a-1}$.*

Proof. It is a direct consequence of Proposition 3.19 and the fact that $h = g^{-1}$. □

Lemma 3.21 ([20, Lemma 5.1])**.** *For all $a \in \mathbf{No}$, $|g(a)|_{+-} \leq |a|_{+-} + 1$.*

Lemma 3.22 ([2, Proposition 3.1])**.** *For all $a \in \mathbf{No}$, $|h(a)|_{+-} \leq \omega^{|a|_{+-}+1}$*

Contrary to the previous statements, the two following lemmas are original up to our knowledge. They contribute to a better understanding of the function g and the comparison between the length of a surreal number and the one of its exponential.

Lemma 3.23. *Let c be a surreal number. Assume $g(a) < c$ for all $a \sqsubset \omega^c$ such that $0 < a < \omega^c$. Then $g(\omega^c) = c_+$ if c does not have any longest prefix greater than itself, otherwise, $g(\omega^c) = c''$ where c'' is the longest prefix of c such that $c'' > c$.*

Proof. By induction on c:

- For $c = 0$, $g(\omega^0) = g(1) = 1$ whose signs sequence is indeed the one of 0 followed by a plus.
- Assume the property for $b \sqsubset c$. Assume $g(a') < c$ for all $a' \sqsubset \omega^c$ such that $0 < a' < \omega^c$. Then,
$$g(\omega^c) = [c \mid g(a'')]$$
where a'' ranges over the elements such that $a'' \sqsubset \omega^c$ and $a'' > \omega^c$.
  - First case: c has a longest prefix c_0 such that $c_0 > c$. Let a'' such that $a'' \sqsubset \omega^c$ and $a'' > \omega^c$. Let c'' such that $a'' \asymp \omega^{c''}$. We necessarily have $c'' \sqsubset c$ and $c'' > c$. Therefore, $c'' \sqsubset c_0$ and $c'' > c_0$. Thus
$$c < c_0 < c'' < g(a'')$$
The simplicity property ensures that $g(\omega^c)$ is a prefix of all surreal x such that $c < x < g(a'')$ for all a'' such that $a'' \sqsubset \omega^c$ and $a'' > \omega^c$. Hence,
$$g(\omega^c) \sqsubseteq c_0 \sqsubset c$$
For any $a < \omega^{c_0}$ such that $a \sqsubset \omega^{c_0}$, we also have $a \sqsubset \omega^c$ and $a < \omega^c$. Thus $g(a) < c < c_0$. We then can apply the induction hypothesis on ω^{c_0} and get one of the following cases
    * If c_0 as a longest prefix c_1 such that $c_1 > c_0$, then $g(\omega^{c_0}) = c_1$. In terms of signs sequences, we have some ordinals α and β such that
$$c_0 = (c_1)(-)(+)^\alpha \qquad \text{and} \qquad c = (c_0)(-)(+)^\beta$$
Thus $$c_0 = [c \mid c_1]$$
But, by definition $c < g(\omega^c) < g(\omega^{c_0}) = c_1$. Thus, by the simplicity property, $c_0 \sqsubseteq g(\omega^c)$. Finally, $g(\omega^c) = c_0$.
    * If c_0 has no longest prefix c_1 such that $c_1 > c_0$, then $g(\omega^{c_0}) = (c_0)_+$. In terms of signs sequences, we have some ordinal α such that
$$c = (c_0)(-)(+)^\alpha$$
Thus $$c_0 = \left[c \;\middle|\; (c_0)_+\right]$$
But, by definition $c < g(\omega^c) < g(\omega^{c_0}) = (c_0)_+$. Thus, by the simplicity property, $c_0 \sqsubseteq g(\omega^c)$. Finally, $g(\omega^c) = c_0$.
  - Second case: c does not have a longest prefix greater than c. Then,
$$g(\omega^c) = \left[c \;\middle|\; g(\omega^{c''})\right]$$

where c'' ranges over the prefixes of c greater than c. Let $d \sqsubset c$ such that $d > c$. Then there is d_1 of minimal length such that $d \sqsubset d_1 \sqsubset c$ and $d_1 > c$. By minimality of d_1, d is the longest prefix of d_1 greater than d_1. As in the first case, we can apply the induction hypothesis on d_1 and get $g(\omega^{d_1}) = d$. Thus all the prefixes c greater than c appear in the elements $g(\omega^{c''})$ for c'' a prefix of c greater than c. The only other possible value of $g(\omega^{c''})$ is d_+ for some d a prefix of c greater than c. Hence it has no effect in the computation of $g(\omega^c)$. Finally,

$$g(\omega^c) = \left[c \;\middle|\; c'', c''_+\right] = [c \mid c'']$$

where c'' ranges over the prefixes of c greater than c. We finally conclude that $g(\omega^c) = c_+$.

□

Lemma 3.24. *For all $a > 0$, $|a|_{+-} \leq \left|\omega^{g(a)}\right|_{+-} \otimes (\omega + 1)$.*

Proof. We proceed by induction on $|a|_{+-}$.

- For $a = 1$, $g(a) = 1$ and we indeed have $1 \leq \omega^2 + \omega$.
- Assume the property for all $b \sqsubset a$. Let c such that $\omega^c \asymp a$. Then

$$g(a) = [c, g(a') \mid g(a'')]$$

  We split into two cases:
  - If there is some $a_0 \sqsubset a$ such that $a_0 < a$ and $g(a_0) \geq c$ then

$$g(a) = [g(a') \mid g(a'')]$$

    Let S the signs sequence such that the signs sequence of a is the signs sequence of a_0 followed by S. By an easy induction on the length of S, we can show that the signs sequence of $g(a)$ is the signs sequence of $g(a_0)$ followed S. Let α the length of S. Therefore using Theorem 3.9,

$$\left|\omega^{g(a)}\right|_{+-} \geq \left|\omega^{g(a_0)}\right|_{+-} \oplus (\omega \otimes \alpha)$$

    and then,

$$\begin{aligned}
\left|\omega^{g(a)}\right|_{+-} \otimes (\omega+1) &\geq \left(\left|\omega^{g(a_0)}\right|_{+-} \oplus (\omega \otimes \alpha)\right) \otimes (\omega+1) \\
&\geq \left(\left|\omega^{g(a_0)}\right|_{+-} \oplus (\omega \otimes \alpha)\right) \otimes \omega \oplus \left|\omega^{g(a_0)}\right|_{+-} \oplus (\omega \otimes \alpha) \\
&\geq \left|\omega^{g(a_0)}\right|_{+-} \otimes \omega \oplus \left|\omega^{g(a_0)}\right|_{+-} \oplus (\omega \otimes \alpha) \\
&\geq \left|\omega^{g(a_0)}\right|_{+-} \otimes (\omega+1) \oplus (\omega \otimes \alpha)
\end{aligned}$$

    and by induction hypothesis on a_0,

$$\left|\omega^{g(a)}\right|_{+-} \otimes (\omega+1) \geq |a_0|_{+-} \oplus (\omega \otimes \alpha) \geq |a_0|_{+-} \oplus \alpha = |a|_{+-}$$

  - Otherwise, for any $a_0 \sqsubset a$ such that $a_0 < a$, $g(a_0) < c$. Therefore,

$$g(a) = [c \mid g(a'')]$$

    Also, since $a > 0$, we can write the signs sequence of a as the one of ω^c followed by some signs sequence S. If S contains a plus, then there is a prefix of a, a_0 such that $a_0 < a$ and still $a_0 \asymp \omega^c$ and then $g(a_0) > c$ what is not the case by assumption. Then, S is a sequence of minuses. Let α be the length of S. Again, by an easy induction on α, the signs sequence of $g(a)$ is the one of $g(\omega^c)$ followed by S. Hence,

$$\left|\omega^{g(a)}\right|_{+-} \geq \left|\omega^{g(\omega^c)}\right|_{+-} \oplus (\omega \otimes \alpha)$$

If S is not the empty signs sequence, as in the previous case but using the induction hypothesis on ω^c,

$$\left|\omega^{g(a)}\right|_{+-} \otimes (\omega + 1) \geq |\omega^c|_{+-} \oplus \alpha = |a|_{+-}$$

Now if S is the empty sequence, $a = \omega^c$. Applying Lemma 3.23 to c we get that either $g(a) = c_+$ or $g(a)$ is the last prefix of c greater than c. If the first case occurs then a is a prefix of $\omega^{g(a)}$ and then $\left|\omega^{g(a)}\right|_{+-} \geq |a|_{+-}$. Now assume that the second case occurs. Then for any b such that $g(a) \sqsubset b \sqsubset c$, $b < c$. If for all $b' \sqsubset b$ such that $b' < b$, $g(b') < b$, then Lemma 3.23 applies. Since b has a last prefix greater than itself, $g(a)$, $g(\omega^b) = g(a)$ and we reach a contradiction since $b < c$ and therefore $\omega^b < \omega^c = a$. Then for all b such that $g(a) \sqsubset b \sqsubset c$, there is some $b' \sqsubset b$, $b' < b$ such that $g(\omega^{b'}) > b$. Since the signs sequence of b consists in the one of $g(a)$ a minus and then a bunch of pluses, and since $g(\omega^{b'})$ must also a a prefix of c, $g(\omega^{b'}) \sqsubseteq g(a) \sqsubset b$. Therefore to ensure $g(b') > b$, we must have $g(\omega^{b'}) \geq g(a)$. Since $\omega^{b'}$ is a prefix of a lower than a, it is a contradiction. Therefore, there is no b such that $g(a) \sqsubset b \sqsubset c$ and $b < c$, and finally, the signs sequence of c is the one $g(a)$ followed by a minus. In particular, $g(a)$ and c have the same amount of pluses, say α. Then, using Theorem 3.9,

$$\begin{aligned}
|a|_{+-} &= \left|\omega^{g(a)}\right|_{+-} \oplus \omega^{\alpha+1} \\
&\leq \left|\omega^{g(a)}\right|_{+-} \oplus \left|\omega^{g(a)}\right|_{+-} \otimes \omega = \left|\omega^{g(a)}\right|_{+-} \otimes \omega \\
&\leq \left|\omega^{g(a)}\right|_{+-} \otimes (\omega + 1)
\end{aligned}$$

We conclude using the induction principle.

□

Corollary 3.25. *For all $a > 0$ and for all multiplicative ordinal greater than ω, if $|a|_{+-} \geq \mu$, then $\left|\omega^{g(a)}\right|_{+-} \geq \mu$.*

Proof. Assume the that $\left|\omega^{g(a)}\right|_{+-} < \mu$. Then using Lemma 3.24,

$$\mu \leq \left|\omega^{g(a)}\right|_{+-} \otimes (\omega + 1)$$

Since μ is a multiplicative ordinal greater than ω, we have $\omega + 1 < \mu$. μ is a multiplicative ordinal, hence $\left|\omega^{g(a)}\right|_{+-} \otimes (\omega + 1) < \mu$ and we reach a contradiction.

□

3.3. **Paths.** In the previous subsection, we saw that for all surreal numbers in Conway normal form $a = \sum_{i<\nu} r_i\omega^{a_i}$, we have

$$\ln\omega^a = \sum_{i<\nu} r_i\omega^{h(a_i)}$$

and if a is purely infinite, $\exp x = \omega^{\sum_{i<\nu} r_i\omega^{g(a_i)}}$

This means that h and g are in fact reciprocal that we can in fact write a in the following way

$$a = \sum_{i<\nu} r_i \exp(x_i)$$

where for $i<\nu$ $x_i = \ln\omega^{a_i}$

Using this we can do the same thing for all the x_is giving thus a tree structure. We could have done this also to the a_is giving another tree but the function $x \mapsto \omega^x$ may be less convenient to work with since $x<y \implies \omega^x \prec \omega^y$ and thus there is no hope for any "continuity" (for any reasonable notion of "continuity").

Definition 3.26. Let x be a surreal number. A **path** P of x is sequence $P:\mathbb{N}\to\mathbf{No}$ such that

- $P(0)$ is a term of x
- For all $i\in\mathbb{N}$, $P(i+1)$ is an infinite term of $\ln|P(i)|$

We denote $\mathcal{P}(x)$ the set of all paths of x.

Definition 3.27 (Log-atomic)**.** A positive surreal number $x\in\mathbf{No}_+^*$ is said **log-atomic** *iff* for all $n\in\mathbb{N}$, there is a surreal number a_n such that $\ln_n x = \omega^{a_n}$. We denote $\mathbb{L}$ for the class of log-atomic numbers.

Remark. Actually, all the a_ns can be taken positive since if $a_n \le 0$ then $\ln_{n+1} x \le 0$ which is impossible because of the existence of a_{n+1}.

Example. A typical example is ω. We can check that for all $n\in\mathbb{N}$, $\ln_n\omega = \omega^{\frac{1}{\omega^n}}$. Therefore

$$\{\exp_n\omega, \ln_n\omega \mid n\in\mathbb{N}\} \subseteq \mathbb{L}$$

4. Fields closed under the exponential and logarithmic functions

This section is dedicated to the proofs of Theorems 1.3, 1.4 and 1.5 together with Proposition 1.6.

4.1. **An instability result.** Recall that in [20, Proposition 4.7], we are provided the following decomposition:

$$\mathbf{No}_\lambda = \bigcup_\mu \mathbb{R}_\lambda^{\mathbf{No}_\mu}$$

This decomposes $\mathbf{No}_\lambda$ into an increasing hierarchy of subfield. All of these are well described in terms of Hahn series. However, none of these are closed under exponential and logarithm which is stated in the following theorem:

Proposition 1.6. *$\mathbb{R}_\lambda^{\mathbf{No}_\mu}$ is never closed under exponential function for $\mu<\lambda$ a multiplicative ordinal.*

Proof. If μ is a multiplicative ordinal but not an ε-number, $\mu = \omega^{\omega^\alpha}$ for some ordinal $\alpha < \mu$. Since $g(\omega^\alpha) \geq \omega^\alpha$ (Proposition 3.18), we have $\omega^{g(\alpha)} \geq \omega^{\omega^\alpha} = \mu$. In particular, $\omega^{g(\alpha)} \notin \mathbf{No}_\mu$. Moreover, Proposition 3.14 ensures that $\exp(\omega^\alpha) = \omega^{\omega^{g(\alpha)}}$. Therefore, $\exp(\omega^\alpha) \notin \mathbb{R}_\lambda^{\mathbf{No}_\mu}$.
Now, if μ is an ε-number. Take $x = \sum\limits_{0<i<\mu} \omega^{\omega^{-i}}$. Then by Propositions 3.14 and 3.19 we know that

$$\exp(x) = \omega^{\sum\limits_{0<i<\mu} \omega^{g(\omega^{-i})}} = \omega^{\sum\limits_{0<i<\mu} \omega^{-i}}$$

Since μ is an ε-number, for $i < \mu$, $\omega^{-i} \in \mathbf{No}_\mu$ but $\sum\limits_{0<i<\mu} \omega^{-i} \notin \mathbf{No}_\mu$ (as a consequence of Theorem 3.9, the series having length μ, the length of the surreal number is at least μ). Therefore $x \in \mathbb{R}_\lambda^{\mathbf{No}_\mu}$ and $\exp x \notin \mathbb{R}_\lambda^{\mathbf{No}_\mu}$. □

4.2. **Hierarchy of fields closed under exponential and logarithm.** To get fields in terms of Hahn series that are closed under exponential and logarithm, we first characterise when a union of Hahn field gets the sought property. This is the purpose of the following proposition:

Proposition 4.1. *Let λ be an ε-number and $(\Gamma_i)_{i\in I}$ be a family of Abelian subgroups of* **No**. *Then $\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is closed under* exp *and* ln *if and only if*

$$\bigcup_{i\in I} \Gamma_i = \bigcup_{i\in I} \mathbb{R}_\lambda^{g\left((\Gamma_i)_+^*\right)}$$

Proof. $\overset{\text{NC}}{\Rightarrow}$ We assume that $\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is closed under both exponential and logarithm. Then for any $x = \sum\limits_{i<\nu} r_i \omega^{a_i}$ a purely infinite number, using Proposition 3.14, we have

$$\exp x = \omega^{\sum\limits_{i<\nu} r_i \omega^{g(a_i)}} \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$$

and therefore
$$\sum_{i<\nu} r_i \omega^{g(a_i)} \in \bigcup_{j\in I} \Gamma_j$$

This being true for any family $(a_i)_{i<\nu}$ of Γ_j, for any $j \in I$. Hence,

$$\bigcup_{i\in I} \mathbb{R}_\lambda^{g\left((\Gamma_i)_+^*\right)} \subseteq \bigcup_{i\in I} \Gamma_i$$

Conversely, for any $j \in I$ and any $a \in \Gamma_j$ then we have $\ln \omega^a \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$. Writing $a = \sum\limits_{i<\nu} r_i \omega^{a_i}$, we get $\sum\limits_{i<\nu} r_i \omega^{h(a_i)} \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$. To say it another way,

$$\exists k \in I \quad \forall i < \nu \qquad h(a_i) \in \Gamma_k$$

or
$$\exists k \in I \quad \forall i < \nu \qquad a_i \in g\left((\Gamma_k)_+^*\right)$$

Then, there is some $k \in I$ such that $a \in \mathbb{R}_\lambda^{g\left((\Gamma_k)_+^*\right)}$. Hence, for all $j \in I$, $\Gamma_j \subseteq \bigcup\limits_{i\in I} \mathbb{R}_\lambda^{g\left((\Gamma_i)_+^*\right)}$. Finally,

$$\bigcup_{i\in I} \mathbb{R}_\lambda^{g\left((\Gamma_i)_+^*\right)} \supseteq \bigcup_{i\in I} \Gamma_i$$

Having both inclusions, we get

$$\bigcup_{i\in I}\mathbb{R}_\lambda^{g\left((\Gamma_i)_+^*\right)} = \bigcup_{i\in I}\Gamma_i$$

$\overset{\text{SC}}{\Leftarrow}$ We assume that $\bigcup_{i\in I}\mathbb{R}_\lambda^{g\left((\Gamma_i)_+^*\right)} = \bigcup_{i\in I}\Gamma_i$. We split the proof into several steps

(i) First take $x = \sum_{i<\nu} r_i\omega^{a_i} \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ being appreciable, i.e. $a_i \leq 0$ for all $i < \nu$. By definition there is some $j \in I$ such that x is an element of $\mathbb{R}_\lambda^{\Gamma_j}$. Following Theorem 3.13,

$$\operatorname{supp}\exp x \subseteq \langle \operatorname{supp} x\rangle$$

where $\langle \operatorname{supp} x\rangle$ is the monoid generated by $\operatorname{supp} x$ in Γ_j. In particular $\operatorname{supp}\exp x \subseteq \Gamma_j$. Then, Proposition 2.8 ensures that the order type of $\operatorname{supp}\exp x$ is less than λ. Hence $\exp x \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$.

(ii) Let $x = \sum_{i<\nu} r_i\omega^{a_i} \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ a purely infinite number. Let $j \in I$ such that $x \in \mathbb{R}_\lambda^{\Gamma_j}$ that is that $a_i \in (\Gamma_j)_+^*$ for all $i < \nu$. We have $\exp x = \omega^{\sum_{i<\nu} r_i\omega^{g(a_i)}}$ and

$$\sum_{i<\nu} r_i\omega^{g(a_i)} \in \mathbb{R}_\lambda^{g\left((\Gamma_j)_+^*\right)}$$

By assumption, $\mathbb{R}_\lambda^{g\left((\Gamma_j)_+^*\right)} \subseteq \bigcup_{i\in I}\Gamma_i$. Then $\exp x \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$.

(iii) We now use both Items (i) and (ii). Let $x \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ be arbitrary. Let x_∞ be its purely infinite part and x_a its appreciable part. Then $x = x_\infty + x_a$ and $\exp x = \exp(x_\infty)\exp(x_a)$. Using (ii) and (i) respectively, we have $\exp x_\infty \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ and $\exp x_a \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$. Then since $\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is a field, $\exp x \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$.

(iv) Similarly to Point (i), if $x = \sum_{i<\nu} r_i\omega^{a_i} \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is infinitesimal, i.e. $a_i < 0$ for all $i < \nu$, then

$$\ln(1+x) = \sum_{k=1}^{\infty}\frac{x^k}{k} \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$$

(v) Let $a \in \bigcup_{i\in I}\Gamma_i$. By assumption there is $j \in I$ such that $a \in \mathbb{R}_\lambda^{g\left((\Gamma_j)_+^*\right)}$. Hence, we can write $a = \sum_{i<\nu} r_i\omega^{g(a_i)}$ where $\nu < \lambda$ and $a_i \in (\Gamma_j)_+^*$ for all $i < \nu$. Then, $\ln\omega^a = \sum_{i<\nu} r_i\omega^{a_i}$. Hence $\ln\omega^a \in \mathbb{R}_\lambda^{\Gamma_j} \subseteq \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$.

(vi) Let $x \in \left(\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}\right)_+^*$ be arbitrary and write it as $x = r\omega^a(1+\varepsilon)$ where ε is infinitesimal, r is a positive real number and a a surreal number. Then, $\ln x = \ln\omega^a + \ln r + \ln(1+\varepsilon)$. Then since $\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is a field, $\exp x \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$. Using (v) and (iv) respectively, we have $\ln\omega^a \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ and $\ln(1+\varepsilon) \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$. Then, since $\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is a field containing $\mathbb{R}$, $\ln x \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$.

Item (iii) proves that $\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is closed under exponential and Item (vi) that $\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is closed under logarithm. This is what was announced. □

Corollary 4.2. *Let λ be an ε-number and Γ be an Abelian subgroup of* $\mathbf{No}$. *Then $\mathbb{R}_\lambda^\Gamma$ is closed under* exp *and* ln *if and only if $\Gamma = \mathbb{R}_\lambda^{g\left(\Gamma_+^*\right)}$.*

This result is quite similar to Theorem 1.3 but in the very particular case where $\bigcup_{G\in\Gamma^{\uparrow\lambda}} G = \Gamma$. This applies for instance when $\Gamma = \{0\}$. In this case, we get $\mathbb{R}_\lambda^\Gamma = \mathbb{R}$. If λ is a regular cardinal we get another example considering $\mathbb{R}_\lambda^\Gamma = \Gamma = \mathbf{No}_\lambda$ which is a result that is very similar to the result of Kuhlmann and Shelah in [15].
Using the previous proposition, we can now prove Theorem 1.3.

Theorem 1.3. *Let Γ be an Abelian subgroup of* $\mathbf{No}$ *and λ be an ε-number, then $\mathbb{R}_\lambda^{\Gamma^{\uparrow\lambda}}$ is closed under exponential and logarithmic functions.*

Proof. We write $\Gamma^{\uparrow\lambda} = (\Gamma_\beta)_{\beta<\gamma_\lambda}$. Using Proposition 4.1, we just need to show

$$\bigcup_{\beta<\gamma_\lambda} \Gamma_\beta = \bigcup_{\beta<\gamma_\lambda} \mathbb{R}_\lambda^{g\left((\Gamma_\beta)_+^*\right)}$$

- ($\supseteq$) Let $x \in \mathbb{R}_\lambda^{g\left((\Gamma_\beta)_+^*\right)}$. Let $n < \gamma_\lambda$ minimal such that $\nu(x) < e_n$. Then $x \in \Gamma_{\max(n,\beta)}$.
- ($\subseteq$) Let $x \in \Gamma_\beta$. Write $x = \sum_{i<\nu} r_i\omega^{a_i}$. We also have $x = \sum_{i<\nu} r_i\omega^{g(h(a_i))}$ and $h(a_i) \in \Gamma_{\beta+1}$. Then $x \in \mathbb{R}_\lambda^{g\left((\Gamma_{\beta+1})_+^*\right)}$.

□

Now, we can use this statement to decompose $\mathbf{No}_\lambda$ into a hierarchy of fields, all closed under the exponential and logarithmic functions. First, all the fields belonging to the hierarchy must be included in $\mathbf{No}_\lambda$.

Proposition 4.3. *Let λ be an ε-number and $\mu < \lambda$ an additive (or multiplicative) ordinal. If $\Gamma \subseteq \mathbf{No}_\mu$ then $\mathbb{R}_\lambda^{\Gamma^{\uparrow\lambda}} \subseteq \mathbf{No}_\lambda$*

Proof. Write $\Gamma^{\uparrow\lambda} = (\Gamma_\beta)_{\beta<\gamma_\lambda}$. What we have to prove is that for all $i < \gamma_\lambda$, $\Gamma_i \subseteq \mathbf{No}_{\mu_i}$ for some $\mu_i < \lambda$. We will even prove that $\mu_i = e_{k\oplus 2\otimes i}$ works for some fixed ordinal k. We prove it by induction on i.

- For $i = 0$, $\mu_0 = e_k$ with k the least ordinal such that $\mu \leq e_k$ works.
- Assume $i = j + 1$ and that the property is true for j. Therefore Γ_i is the group generated by Γ_j, $\mathbb{R}_{e_j}^{g\left((\Gamma_j)_+^*\right)}$ and $\left\{ h(a_k) \;\middle|\; \sum_{k<\nu} r_k\omega^{a_k} \in \Gamma_j \right\}$. Thanks to the induction hypothesis and Lemma 3.21, $g\left((\Gamma_j)_+^*\right) \subseteq \mathbf{No}_{\mu_j}$, since μ_j is an additive ordinal. Hence, thanks to Lemma 3.10, $\mathbb{R}_{e_j}^{g\left((\Gamma_j)_+^*\right)} \subseteq \mathbf{No}_{\omega^{\mu_j\otimes\omega}\otimes e_j}$. Finally, from Lemmas 3.22 and 3.10, $h(a_k) \in \mathbf{No}_{\omega^{\mu_j}}$. Thus, $\Gamma_i \subseteq \mathbf{No}_{\omega^{\mu_j\otimes\omega}\otimes e_j}$. Since $\omega^{\mu_j\otimes\omega}, e_j < e_{k\oplus 2\otimes i}$, and $e_{k\oplus 2\otimes i}$ is multiplicative, taking, $\mu_i = e_{k\oplus 2\otimes i}$ works.
- If $i < \gamma_\lambda$ is a limit ordinal, for all $j < i$, $\lambda > e_{k\oplus 2\otimes i} > e_{k\oplus 2\otimes j}$. Then, by the induction hypothesis on all $j < i$, $\Gamma_i \subseteq \mathbf{No}_{e_{k\oplus 2\otimes i}}$.

□

We are now ready to prove that we indeed have a hierarchy.

Theorem 1.4. *Let λ be an ε-number. $\mathbf{No}_\lambda = \bigcup_\mu \mathbb{R}_\lambda^{{\mathbf{No}_\mu}^{\uparrow\lambda}}$, where μ ranges over the additive ordinals less than λ (equivalently, μ ranges over the multiplicative ordinals less than λ).*

Proof. Recall that [20, Proposition 4.7] states the following:

$$\mathbf{No}_\lambda = \bigcup_{\mu\in\{\,\mu<\lambda\ \mid\ \mu \text{ additive ordinal}\}} \mathbb{R}_\lambda^{\mathbf{No}_\mu}$$

By definition of $\mathbb{R}_\lambda^{{\mathbf{No}_\mu}^{\uparrow\lambda}}$, it must contain $\mathbb{R}_\lambda^{\mathbf{No}_\mu}$ and then

$$\mathbf{No}_\lambda \subseteq \bigcup_{\mu\in\{\,\mu<\lambda\ \mid\ \mu \text{ additive ordinal}\}} \mathbb{R}_\lambda^{{\mathbf{No}_\mu}^{\uparrow\lambda}}$$

On the other hand, applying Proposition 4.3 gives

$$\bigcup_{\mu\in\{\,\mu<\lambda\ \mid\ \mu \text{ additive ordinal}\}} \mathbb{R}_\lambda^{{\mathbf{No}_\mu}^{\uparrow\lambda}} \subseteq \mathbf{No}_\lambda$$

and this concludes the proof. □

We finally prove that the hierarchy is strict, making it interesting in itself as none of the fields involved are equal to the global field. To prove that, we first claim that the construction of $\Gamma^{\uparrow\lambda}$ does not create new log-atomic numbers (up to some iteration of exp or ln). After that, we will prove that we do introduce new log-atomic numbers when going through the hierarchy.

Lemma 4.4. *Write $\Gamma^{\uparrow\lambda} = (\Gamma_\beta)_{\beta<\gamma_\lambda}$, and let*

$$L = \left\{\exp_n x, \ln_n x \;\middle|\; x\in\mathbb{L},\quad n\in\mathbb{N},\quad \exists y\in\mathbb{R}_\lambda^{\Gamma}\ \exists P\in\mathcal{P}(y)\ \exists k\in\mathbb{N}\quad P(k)=x\right\}$$

we have for all $i<\gamma_\lambda$,

$$L = \left\{\exp_n x, \ln_n x \;\middle|\; x\in\mathbb{L},\quad n\in\mathbb{N},\quad \exists y\in\mathbb{R}_\lambda^{\Gamma_i}\ \exists P\in\mathcal{P}(y)\ \exists k\in\mathbb{N}\quad P(k)=x\right\}$$

Proof. We prove it by induction on i.

- For $i=0$, $\Gamma_0=\Gamma$ then there is noting to prove.
- Assume the property for some ordinal $i<\gamma_\lambda$. We prove it for $i+1$.
  - ($\subseteq$) Trivial since $\mathbb{R}_\lambda^{\Gamma_i}\subseteq\mathbb{R}_\lambda^{\Gamma_{i+1}}$.
  - ($\supseteq$) Let $x\in\mathbb{L}$, $y\in\mathbb{R}_\lambda^{\Gamma_{i+1}}$, $P\in\mathcal{P}(y)$ and $k\in\mathbb{N}$ such that $P(k)=x$. Write $P(0)=r\omega^a$ a term of x with $a\in\Gamma_{i+1}$. Then a can be written

$$a = u+v+\sum_{j=1}^{k}\sigma_j h(w_j)$$

with $u\in\Gamma_i$, $v\in\mathbb{R}_{e_i}^{g\left((\Gamma_i)_+^*\right)}$, $\sigma_j\in\{-1,1\}$ and $w_j\in\Gamma_i$. By definition of a path, $P(1)$ is a purely infinite term of

$$\ln\omega^a = \ln\omega^u+\ln\omega^v+\sum_{j=1}^{k}\ln\omega^{\sigma_j h(w_j)} = \ln\omega^u+\ln\omega^v+\sum_{j=1}^{k}\sigma_j\ln\omega^{h(w_j)}$$

Then, up to a real factor s, $P(1)$ is a term of either $\ln \omega^u$ or $\ln \omega^v$ or $\ln \omega^{h(w_j)}$ for some j.

- ∗ Case 1: $sP(1)$ is a purely infinite term of $\ln \omega^u$. Then the function

$$Q(m) = \begin{cases} \omega^u & \text{if } m = 0 \\ sP(1) & \text{if } m = 1 \\ P(m) & \text{if } m > 1 \end{cases}$$

is a path of $\omega^u \in \mathbb{R}_\lambda^{\Gamma_i}$. Then, if $m \geq \max(2, n)$, $Q(m) = P(m) = \ln_{m-n}(x)$ then for all $n \in \mathbb{N}$,

$$\exp_n x, \ln_n x \in \left\{ \exp_n x, \ln_n x \;\middle|\; \begin{array}{c} x \in \mathbb{L}, \quad n \in \mathbb{N}, \\ \exists y \in \mathbb{R}_\lambda^{\Gamma_{i+1}} \; \exists P \in \mathcal{P}(y) \; \exists k \in \mathbb{N} \quad P(k) = x \end{array} \right\}$$

- ∗ Case 2: $sP(1)$ is a purely infinite term of $\ln \omega^v$. Write $v = \sum_{i<\nu'} s_i \omega^{g(b_i)}$ where $b_i \in \Gamma_k$. Again, the function

$$Q(m) = \begin{cases} sP(1) & \text{if } m = 0 \\ P(m+1) & \text{if } m > 0 \end{cases}$$

is a path of $\ln \omega^v = \sum_{i<\nu'} s_i \omega^{b_i} \in \mathbb{R}_\lambda^{\Gamma_i}$. Then,if $m \geq \max(1, n-1)$, $Q(m) = \ln_{m-n+1} x \in L$ and we are done.

- ∗ Case 3: $sP(1)$ is a purely infinite term of $\ln \omega^{h(w_j)}$. From the definition of w_j, there is $s' \in \mathbb{R}^*$ such that $s'\omega^{w_j}$ is a term of some element of $y \in \Gamma_n$. Then $s'\omega^{h(w_i)}$ is a purely infinite term of $\ln \omega^y$. Then the function

$$Q(m) = \begin{cases} \omega^y & \text{if } m = 0 \\ s'\omega^{h(w_i)} & \text{if } m = 1 \\ sP(1) & \text{if } m = 2 \\ P(m-1) & \text{if } m > 2 \end{cases}$$

is a path of $\omega^y \in \mathbb{R}_\lambda^{\Gamma_i}$. Then, if $m \geq \max(3, n+1)$, $Q(m) = \ln_{m-n-1} x \in L$ and we are done.

- Let $i < \gamma_\lambda$ be a limit ordinal. Assume the property for $j < i$. We have that $\Gamma_i = \bigcup_{j<i} \Gamma_j$. Again we just need to prove one inclusion, the other one being trivial. Let $x \in \mathbb{L}$ and $y \in \mathbb{R}_\lambda^{\Gamma_i}$, $P \in \mathcal{P}(y)$ and $n \in \mathbb{N}$ minimal such that $P(n) = x$. Write $P(0) = r\omega^a$ with $a \in \Gamma_i$. Then there is $j < i$ such that $a \in \Gamma_j$. In particular P is a path of $r\omega^a \in \mathbb{R}_\lambda^{\Gamma_j}$. We conclude using the induction hypothesis on j.

□

Corollary 4.5. *Let Γ be an Abelian additive subgroup of* **No** *and*

$$L = \left\{ \exp_n x, \ln_n x \;\middle|\; x \in \mathbb{L}, \quad n \in \mathbb{N}, \quad \exists y \in \mathbb{R}_\lambda^{\Gamma} \; \exists P \in \mathcal{P}(y) \; \exists k \in \mathbb{N} \quad P(k) = x \right\}$$

Then,

$$L = \left\{ \exp_n x, \ln_n x \;\middle|\; \begin{array}{c} x \in \mathbb{L}, \quad n \in \mathbb{N}, \\ \exists y \in \mathbb{R}_\lambda^{\Gamma^{\uparrow\lambda}} \; \exists P \in \mathcal{P}(y) \; \exists k \in \mathbb{N} \quad P(k) = x \end{array} \right\}$$

Proof. Just apply the definition of $\mathbb{R}_\lambda^{\Gamma^{\uparrow\lambda}}$ and Lemma 4.4. □

We now prove the Theorem 1.5.

Theorem 1.5. *For all ε-number λ, the hierarchy in Theorem 1.4 is strict:*

$$\mathbb{R}_\lambda^{\mathbf{No}_\mu{}^{\uparrow\lambda}} \subsetneq \mathbb{R}_\lambda^{\mathbf{No}_{\mu'}{}^{\uparrow\lambda}}$$

for all multiplicative ordinals μ and μ' such that $\omega < \mu < \mu' < \lambda$.

Proof. Let λ be an epsilon number. Let $\mu < \mu' < \lambda$ be multiplicative ordinals. Let $x = \omega^{\omega^{-\mu}}$. Clearly, $x \in \mathbb{R}_\lambda^{\mathbf{No}_{\mu'}} \subseteq \mathbb{R}_\lambda^{\mathbf{No}_{\mu'}{}^{\uparrow\lambda}}$. So we will prove that $x \notin \mathbb{R}_\lambda^{\mathbf{No}_\mu{}^{\uparrow\lambda}}$. Note x is a log-atomic number, indeed using Corollary 3.20 $\ln_n x = \omega^{\omega^{-\mu-n}}$. Then applying Corollary 4.5 to both $\mathbf{No}_\mu$ and $\mathbf{No}_{\mu'}$ we just need to show that

$$x \notin \left\{ \exp_n x, \ln_n x \;\middle|\; x \in \mathbb{L}, \quad n \in \mathbb{N}, \quad \exists y \in \mathbb{R}_\lambda^{\mathbf{No}_\mu} \; \exists P \in \mathcal{P}(y) \; \exists k \in \mathbb{N} \quad P(k) = x \right\}$$

Assume the converse. Then there is some path P such that $P(0) \in \mathbb{R}\omega^{\mathbf{No}_\mu}$ and there is some natural numbers $n, k \in \mathbb{N}$ such that $P(k) = \ln_n x$. We prove by induction on i that for all $i \in [\![0 \; ; \; k]\!]$, $|a_i|_{+-} \geq \mu$ where $P(i) = r_i \omega^{a_i}$,

- For $i = k$, $P(i) = \omega^{\omega^{-\mu-n}}$ and using Theorem 3.9,
$$\left|\omega^{-\mu-n}\right|_{+-} = \omega \otimes (\mu + n) \geq \mu$$
- Assume the property for some $i \in [\![1 \; ; \; k]\!]$. By definition of a path, writing $P(i-1) = r_{i-1}\omega^{a_{i-1}}$ and $a_{i-1} = \sum\limits_{j<\nu} s_j \omega^{b_j}$, there is some $j_0 < \nu$ such that $b_{j_0} = g(a_i)$ and $s_{j_0} = r_i$. Using induction hypothesis and Corollary 3.25, $\left|\omega^{b_{j_0}}\right|_{+-} \geq \mu$ and therefore $\left|s_{j_0}\omega^{b_{j_0}}\right|_{+-} \geq \mu$. Now using Lemma 3.11, $|a_{i-1}|_{+-} \geq \left|s_{j_0}\omega^{b_{j_0}}\right|_{+-} \geq \mu$.

The induction principle can then be used to conclude that $|a_0|_{+-} \geq \mu$. But since $P(0) \in \mathbb{R}\omega^{\mathbf{No}_\mu}$, $|a_0|_{+-} < \mu$. We reach a contradiction. Consequently, $x \notin \mathbb{R}_\lambda^{\mathbf{No}_\mu}$. □

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