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QUENTIN GUILMANT

Composition du Jury :

Joris Van der Hoeven DR CNRS, École Polytechnique (LIX)	Président
Emmanuel Jeandel Professeur à l'Université de Lorraine	Rapporteur
Mickael Matusinski HdR, Maitre de Conférence à Université de Bordeaux	Rapporteur
Nathalie Aubrun CR CNRS, Université PARIS SACLAY	Examinateur
Alessandro Berarducci Professeur, Université de Pise	Examinateur
Olivier Bournez Professeur, École Polytechnique (LIX)	Directeur de thèse

Chapter 1

Introduction

1.1 General introduction

Surreal numbers The initial motivation of this thesis was to study continuous models of computations in the context of surreal numbers. These numbers are quite special as they encapsulate almost everything we would expect from the vague notion of "numbers". To be more precise, we are interested in ordered field. In this context, a number is just an element of such a field.
The intuition of what is a number may begin with the idea of natural numbers and their completion under addition. This forms the ring of integers, $\mathbb{Z}$. However, this is a ring, not a field: it lacks a division. This is solved by the field of rational numbers, $\mathbb{Q}$. However, this field is still a bit odd. It is possible to build a sequence of rationals that converges but not to an element of $\mathbb{Q}$. For instance $(u_n)_{n\in\mathbb{N}}$ defined by induction by

$$u_0 = 1 \qquad \forall n \in \mathbb{N} \quad u_{n+1} = \frac{1}{2+u_n}$$

is a sequence of rational numbers which converges to $\sqrt{2}-1$, an irrational number. If we close the field $\mathbb{Q}$ under taking all these limits, we end up with the Cauchy-completion of $\mathbb{Q}$, the field of real numbers, $\mathbb{R}$. However, $\mathbb{R}$ does not encapsulate the idea that there is something greater than all natural numbers. In fact there is a single (up to isomorphism) ordered field with no such a number: $\mathbb{R}$. Thus, the non-existence of such a large number is much more a particular case than a general one. As an example, Hardy fields may contain many of these elements. The consideration of large numbers leads to the construction of ordinal numbers. These are not included in $\mathbb{R}$, but that still come with a notion of order, which extend the order over integers. Therefore, we can look at the field $\mathbb{R}(\omega)$ where ω is the least infinite ordinal number. This field is ordered and again we can go deeper and deeper, taking bigger and bigger ordinals and taking Cauchy-completions. Eventually, we may end up with a class of all possible numbers, surreal numbers.
The (class-)field of surreal numbers is a universal ordered (class-)field, in which we can embedded any ordered field. They also form a real closed (class-)field and thus are looking good, at first sight, to do some analysis on it. Note that we are not speaking about non-standard analysis since in the context of surreal numbers, there is no *transfer principle* in action. However, as a consequence of quantifier elimination for the theory of real closed fields, any formula about them is still satisfied in the surreal numbers (see for instance [46]). A study of analysis in the context of surreal numbers has been conducted by Alling in 1987 in his book "Foundation of analysis over surreal numbers fields" [1]. Other authors have also contributed such as Fornasiero [22] who tired to get integration for surreal valued functions or Rubinstein-Salzedo and Swaminathan [40] who provided surreal extensions to some usual real functions such as $\arctan$. They also investigated the notion of limit, a notion that Lipparini and Mező also developed [35].
Surreal numbers are a quite recent notion. They were first considered by Conway in the context of Games Theory. He published his classical book "On numbers and games" [18] in 1976 and introduced them with an inductive procedure and the use of equivalence classes. Note that this reference book is not the first occurrence of surreal numbers but is the first book Conway dedicated to them. Two years before, in 1974, Knuth published a novel [30] that use Conway's idea to introduce surreal numbers. However, the main reference[1] about the formal construction of surreal numbers with statements and proofs is due to Gonshor [26]. Gonshor was introduced to surreal numbers by Kruskal who decovered how to extend the exponential function over the surreal numbers. Later, surreal number have been related to transseries which are ordinal-long formal series. His book makes a remarkable work of systematic construction and proofs of basics results that are very useful when dealing with such numbers.
Transseries may be seen as some asymptotic behaviors ([49]), as well as surreal numbers. This link was first stated in van der Hoven's book "Transseries and real diffential algebra" ([29]) and since then has been studied deeper ([5, 8, 7]). The reason why surreal numbers may seem suitable for such a study is that any ordered field can be embedded into it and that they still have a notion of simplicity and a somewhat explicit structure. They are also more flexible than

[1] At least according to the author of this thesis.

transseries. Indeed, transseries need new atoms to speak about non-elementary functions ([10, 11]) whereas surreal numbers can have access to elements that are greater than any tower of exponentials the very same way other numbers are built.

The analog thing Our ambition about the relation between surreal numbers and the continuous models of computation is the fact that surreal numbers may encapsulate the notion of "asymptotics". More precisely, we aim to relate surreal numbers, or at least some interesting subfields of them, to the behaviors of some "dynamical systems", which are basically evolution rules that define a trajectory in a space of possible configurations. Models of computation are often known as discrete systems (Turing machines, cellular automata, arithmetic circuits) but they may also be continuous. The continuous model has been theorized by Shanon in 1941 [43] when he introduced the General Analog Computer (GPAC) and they turned out to be deeply related to (polynomial) differential equations, which are a different kind of dynamical systems. It turns out that it is possible to simulate the discrete world into the continuous one ([15]).
GPACs are built from four buildings blocks which are given by Figure 1.1 below. This block takes an input that is a continuous function of time and outputs the result of the corresponding operation as a continuous function of the time. We have

- a constant generator which gives a real number $k \in \mathbb{R}$.
- an addition unit, which outputs the sum of its inputs.
- a multiplication unit, which outputs the products of its inputs.
- an integration units, which, given an input $u(t)$ and the time t, outputs the primitive of u.

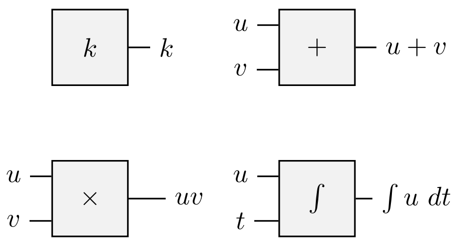


Figure 1.1: Building blocks of a GPAC

All these blocks can be realized by electric circuits or mechanical systems. Actually, the first analog computer were mechanical. Nowadays, it is possible to get their own electronic analog computer[2]. Each of this block must be connected by wires. However, wires must be connected to exactly one output node. Nonetheless, feedback connectors are allowed. To be more precise, we are not allowed to enforce equality by directly connecting two outputs together. It is quite obvious that each node of a GPAC will satisfy a polynomial ordinary differential equation[3]. Therefore, we will use this alternative definition for the characterization of the functions. Using GPACs, the function we can get are divided into two main categories:

Definition 1.1.1 (GPAC-generable function). Let I be an interval of $\mathbb{R}$ containing 0. Let $d \in \mathbb{N}^*$. A function $f = (f_1, \dots, f_d) : I \to \mathbb{R}^d$ is **GPAC-generable** if there are an integer $N \geq d$, polynomials $P_i \in \mathbb{R}[X_1, \dots, X_N]$ and a vector $y_0 \in \mathbb{R}^N$ such that the solution $y = (y_1, \dots, y_N) : I \to \mathbb{R}^N$ to the system

$$\begin{cases} y(0) = y_0 \\ y_i'(t) = P_i(y_1(t), \dots, y_N(t)) \quad t \in I \quad i \in [\![1 \,;\, N]\!] \end{cases}$$

is such that $\forall i \in [\![1 \,;\, d]\!] \qquad f_i = y_i$

Definition 1.1.2 (GPAC-computable function). Let $n, d \in \mathbb{N}^*$ and $I \subseteq \mathbb{R}^n$. A function $f = (f_1, \dots, f_d) : I \to \mathbb{R}^d$ is **GPAC-computable** if there are an integer $N \geq d$, polynomials $P_i \in \mathbb{R}[X_1, \dots, X_N]$ and $Q_i \in \mathbb{R}[X_1, \dots X_n]$ for $i \in [\![1 \,;\, N]\!]$ such that for all $x \in I$, the solution $y = (y_1, \dots, y_N) : I \to \mathbb{R}^N$ to the following polynomial initial value problem:

$$\begin{cases} y_i(0) = Q_i(x) \quad i \in [\![1 \,;\, N]\!] \\ y_i'(t) = P_i(y_1(t), \dots, y_N(t)) \quad t \in I \quad i \in [\![1 \,;\, N]\!] \end{cases}$$

is such that $\exists t_0 > 0 \quad \exists a > 0 \quad \forall t > t_0 \quad \forall i \in [\![1 \,;\, d]\!] \qquad |y_i(t) - f_i(x)| \leq \exp(-at)$

[2]See for instance https://analogparadigm.com/ which sells such computing units.
[3]In fact, if we allow outputs to be connected to each other, we may loose this property.

In other words, a generable function f is such that there is a system that can output $f(t)$ at time t and a computable function is such that there is a system that converges to $f(x)$ as the time grows. We can take a look at the following examples.

Example 1.1.3. The functions $\sin$ and $\cos$ are generable. They are indeed generated by the following system:

$$\begin{cases} y_1(0) = 0 \\ y_2(0) = 1 \end{cases} \qquad \begin{cases} y_1'(t) = y_2(t) \\ y_2(t) = -y_1(t) \end{cases}$$

The associated GPAC is:

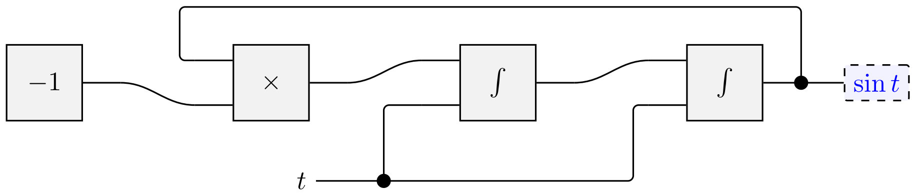


Figure 1.2: A GPAC for $\cos$ and $\sin$.

Example 1.1.4. Euler's gamma function, Γ, defined by

$$\forall x > 0 \qquad \Gamma(x) = \int_0^{+\infty} t^{x-1} \exp(-t)\,\mathrm{d}t$$

is GPAC-computable. To see that we first observe that

$$\begin{aligned} \Gamma(x) &= \int_1^{+\infty} t^{x-1} \exp(-t)\,\mathrm{d}t + \int_0^1 t^{x-1} \exp(-t)\,\mathrm{d}t \\ &= \int_0^{+\infty} (1+t)^{x-1} \exp(-1-t)\,\mathrm{d}t + \int_1^{+\infty} \frac{1}{t^{x+1}} \exp\left(-\frac{1}{t}\right)\,\mathrm{d}t \\ &= \int_0^{+\infty} (1+t)^{x-1} \exp(-1-t)\,\mathrm{d}t + \int_0^{+\infty} \frac{1}{(1+t)^{x+1}} \exp\left(-\frac{1}{1+t}\right)\,\mathrm{d}t \end{aligned}$$

Now, considering the following system:

$$\begin{cases} y_1'(t) &= y_4(t)y_5(t) + y_8(t)y_9(t) \\ y_2'(t) &= 0 \\ y_3'(t) &= -y_3(t)^2 \\ y_4'(t) &= y_2(t)y_3(t)y_4(t) \\ y_5'(t) &= -y_5(t) \\ y_6'(t) &= y_4(t)y_5(t) \end{cases} \quad \begin{aligned} y_7'(t) &= 0 \\ y_8'(t) &= -y_7(t)y_3(t)y_8(t) \\ y_9'(t) &= y_3(t)^2 y_9(t) \\ y_{10}'(t) &= y_8(t)y_9(t) \end{aligned} \qquad \begin{cases} y_2(0) = x-1 & y_7(0) = x+1 \\ y_3(0) = 1 & y_8(0) = 1 \\ y_4(0) = 1 & y_9(0) = e^{-1} \\ y_5(0) = e^{-1} & y_{10}(0) = 0 \\ y_6(0) = 0 & \end{cases}$$

it can be shown that

$$\begin{cases} y_1(t) = y_6(t) + y_{10}(t) = \displaystyle\int_0^t (1+u)^{x-1} \exp(-1-u)\,\mathrm{d}u + \int_0^t \frac{1}{(1+u)^{x+1}} \exp\left(-\frac{1}{1+u}\right)\,\mathrm{d}u \\ y_2(t) = x-1 \qquad\qquad y_7(t) = x+1 \\ y_3(t) = \dfrac{1}{1+t} \qquad\qquad y_8(t) = \dfrac{1}{(1+t)^{x+1}} \\ y_4(t) = (1+t)^{x-1} \qquad\qquad y_9(t) = \exp\left(-\dfrac{1}{1+t}\right) \\ y_5(t) = \exp(-1-t) \qquad\qquad y_{10}(t) = \displaystyle\int_0^t \frac{1}{(1+u)^{x+1}} \exp\left(-\frac{1}{1+u}\right)\,\mathrm{d}u \\ y_6(t) = \displaystyle\int_0^t (1+u)^{x-1} \exp(-1-u)\,\mathrm{d}u \end{cases}$$

In partucular, $y_1(t)$ converges to $\Gamma(x)$. Hence, a GPAC that computes Γ can be drown as follows:

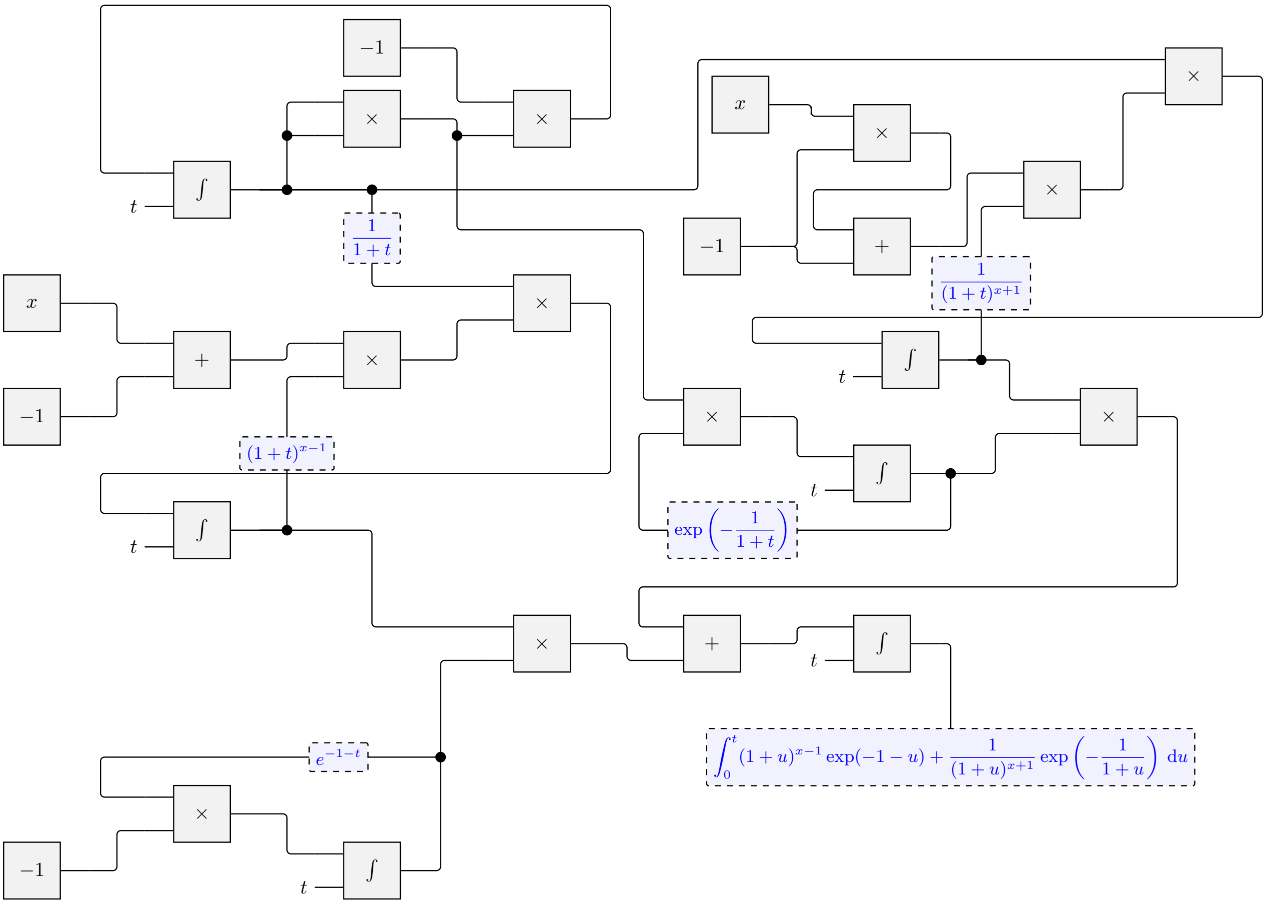


Figure 1.3: A GPAC for Γ.

Further works by Bournez, Compagnolo, Costa, Graça, Hainry or Pouly [44, 15, 38, 16, 14] have made explicit the link between GPAC, solution to polynomial ordinary differential equations and Turing machines on both computability and complexity points of view.

Surreal numbers to describe asymptotic evolution When considering a dynamical system, it is quite confortable to know how it behaves after a long time, in the permanent regime. If we want to tackle this topic with surreal numbers, the notion of differentiation must be investigated so that they can be seen as dynamical system by themselves. This work has been achieved by Berarducci and Mantova ([12]) in 2018. They provided a way to define derivations and anti-derivations over surreal numbers. They also find out a "simplest" derivative, but it turned out that this one is not compatible with any notion of composition over surreal numbers, which is very hard to define in a satisfying way. However, for our purpose, it is not as big as an issue. In fact, "composition" of dynamical systems beyond arithmetic operations is also not well defined. Namely, there is no satisfying general way[4] to get a differential equation for the composition of functions that satisfy (polynomial) ordinary differential equations.
However, surreal number (and transseries), despite being bad at composition, can still handle complicated things such as a primitive of $x \mapsto \exp\left(x^2\right)$. Such a function is known to have no elementary writing. Nevertheless, if we see x as infinitely large number we can find an expression that would play the role of such a function. Indeed, if we derive formally the expression

$$\frac{\exp\left(x^2\right)}{2x}\sum_{k=0}^{+\infty}\frac{(2k)!}{4^k k!}\frac{1}{x^{2k}} \tag{$*$}$$

we end up with $\exp(x^2)$. The problem is that the power series $\sum_{k=0}^{+\infty}\frac{(2k)!}{4^k k!}x^k$ has radius of convergence 0, therefore, we cannot use this expression to evaluate the primitive of $\exp(x^2)$. However, with such an expression, we can handle properly this primitive. Again, we insist on the fact that this expression is not the expression of a function but is still a useful writing to describe an asymptotic behavior of the primitive of $\exp(x^2)$. In fact, in can be shown that if f is a

[4] At least to the knowledge of the author of this thesis

primitive of $\exp(x^2)$, then for all natural number N,

$$f(x) = \frac{\exp\left(x^2\right)}{2x}\sum_{k=0}^{N}\frac{(2k)!}{4^k k!}\frac{1}{x^{2k}} + \underset{x\to+\infty}{O}\left(\frac{\exp(x^2)}{x^{2N+3}}\right)$$

This mean that the writing $(*)$ must be seen has the asymptotic development of a primitive of $\exp(x^2)$ rather than an actual primitive. And this is why it can be considered as the primitive of $\exp(x^2)$ in the context of surreal numbers (or transseries): surreal numbers have much more to do with asymptotic behavior.
Note also that such a primitive is not just an thought experiment: it can be generated by a GPAC. Here is one which performs this computation:

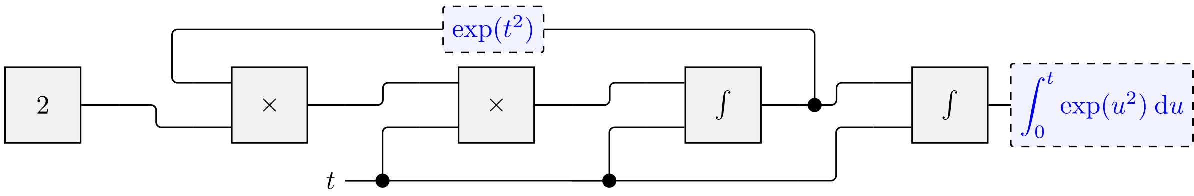


Figure 1.4: A GPAC to generate $x \mapsto \exp(x^2)$ and its primitive.

1.2 Brief and informal state of the art about surreal numbers

The most celebrated way to construct surreal numbers is due to Gonshor. He provided a notation as signs sequences of ordinal length to describe them. The main idea is that we are given a line with a starting point and then walk on it. If we read a $+$, we go one "step" to the right and if we see a $-$, we go to the left. The steps have to be smaller and smaller like if we were performing a dichotomic search. We can illustrate the finite case with the following figure:

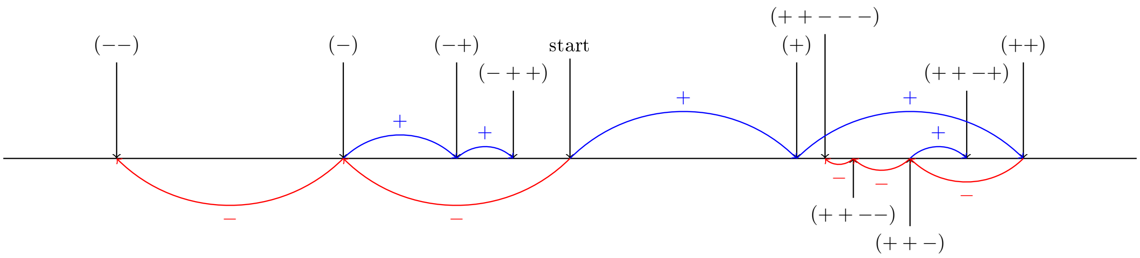


Figure 1.5: The signs sequences on the surreal line.

The formal definition is given by Definition 3.1.3.
With such a notion, it is quite easy to define a concept of simplicity over surreal number. A surreal number x is simpler than y if x is a prefix of y. For instance, $(+--)(+)^\omega$ is simpler than $(+--)(+)^\omega(-)^\omega$. This notion of simplicity defines a well partial order over surreal number, which is quite intuitive since it is closely related to their lengths, which are ordinal numbers.
We can also define a natural order on surreal number, which is just defined by the lexicographic order (up to adding a blank symbol $\square$ such that $- < \square < +$ to compare sequences that have different lengths). That considered, despite being simpler than $(+--)(+)^\omega(-)^\omega$, $(+--)(+)^\omega$ is larger than it.
If we are given two sets of surreal numbers L and R such that each element of L is lower than any element of R, we can wonder what is the simplest surreal number which is both greater than any element of L and smaller than any element of R. We write it $[L \mid R]$. It turns out that such an element always exists. Moreover, fixing a surreal number x, if L is the set of the prefixes of x smaller than x, and R is the set of prefixes of x greater than x, then $x = [L \mid R]$. We call this writing the canonical representation of x. Thanks to this observation, we can write any surreal number x as $x = [L \mid R]$ for some sets L and R.
That being defined, we can make use of it to define operations over surreal numbers. A function f (possibly with several variables) over the surreal numbers is said to be genetic if, $f(x_1, ..., x_n) = [L \mid R]$ where L and R are defined (in the sense that they are definable in the sense of Set Theory) from the sets given by the canonical representations of

$x_1, \ldots, x_n$ and (possibly) $f(y_1, \ldots, y_n)$ where y_i is either x_i or one of its prefixes and at least one of the y_is is a prefix of the corresponding x_i. For instance, we can define addition in a genetic way. If $x = [L_1 \mid R_1]$ and $y = [L_2 \mid R_2]$, we have $x + y = [L \mid R]$ with

$$L = \left\{ x' + y, x + y' \;\middle|\; \begin{array}{c} x' \in L_1 \\ y' \in L_2 \end{array} \right\} \qquad \text{and} \qquad R = \left\{ x'' + y, x + y'' \;\middle|\; \begin{array}{c} x'' \in R_1 \\ y'' \in R_2 \end{array} \right\}$$

It is possible to show that this indeed defines an addition over the surreal numbers with the neutral element, 0, being the empty sequence. It is also possible to define all the field operations in a similar way, even if it is more complicated (see Section 3.2). With such properties we can easily see that there is an embedding of $\mathbb{Z}$ into the surreal numbers (the finite sequences consisting in solely pluses or solely minuses). Therefore, thanks to the division, it is also possible to embed $\mathbb{Q}$ into the surreal numbers. One step further, we also embed $\mathbb{R}$. Indeed, a non rational real number r can be seen as $[\{q \in \mathbb{Q} \mid q < r\} \mid \{q \in \mathbb{Q} \mid q > r\}]$.
Gonshor [26] provided a genetic definition for an exponential function over the surreal numbers (see Section 3.6). This function behaves exactly as expected over real numbers and then is an extension of the usual exponential function over $\mathbb{R}$. This function comes together with its compositional inverse, which is of course an extension of $\ln$ to all positive surreal numbers (see Section 3.6).
Finally, surreal numbers can be seen as transseries (see Sections 4.2 and 4.3) and thus can be understood as dynamics by themselves. Speaking about dynamics, surreal numbers admit a formal derivation (see Section 3.9) which has been developed by Berarducci and Mantova ([12, 13]). Unlike what we mentioned earlier, the definitions of a derivation and the corresponding anti-derivation are not genetic. However, there is still a notion of simplicity since they showed that among all the possible derivations, one is "simpler" than the other. However, the simplest derivation is still not completely satisfying. Indeed, they showed that no suitable notion of composition can make the chain rule ($(f \circ g)' = g' \times f' \circ g$) valid in the context of surreal numbers.
Note that the derivation over surreal numbers is a derivation of surreal numbers themselves and not of functions over surreal number. A lot of attempts have been done to get genetic definition of derivation and anti-derivation for functions defined on surreal numbers (see for instance [22, 40, 1]. However every definition of such an integration fails on very simple and expected properties. For instance, we may not have the uniqueness of a primitive up to some constant factor or we may not be able to satisfy $\displaystyle\int_a^b f = F(b) - F(a)$ when F is a primitive of f (especially because of the previous observation).

1.3 Overview of the contributions and organization of the thesis

In this monograph, we recall the theory the field **No** of all surreal numbers and contribute to establish a working context to the study of surreal numbers in smaller fields than **No** that still have a lot of stability properties. These fields are the foundation of our long term ambition to use surreal numbers and an extension of them to study dynamics of GPACS, hence to study the dynamics of algorithms.
This thesis is organized as follows:

- Chapter 2 is a background chapter that provides basic knowledge about well orders and order types. Nothing is unknown from the scientific community in this chapter but we still provide proofs of some theorems that are hard to find in the literature (this includes Propositions 2.3.6, 2.4.2 and 2.4.3).

- Chapter 3 is the longest part of this monograph. It presents synthetic view of what has been done with the surreal numbers. This chapter is mostly a survey about surreal number but still contains original contributions such as the proofs of some upper bound linked to the derivatives of surreal numbers.

- Chapter 4 makes the link between surreal number and transseries as the later structure is quite well studied in the literature and asymptotic studies.

- Chapter 5 may be the most important and original part of this thesis. In this chapter we show how to build fields of surreal numbers that are stable under almost all the operations we need to study differential equations (exponential, logarithm, derivative, anti-derivative). These constructions may give access to some hierarchy of surreal fields. We think that this hierarchy is a promising approach to study surreal numbers for computation compare to an approach that would consider the whole proper class of surreal numbers.

- Chapter 6 introduces some topological considerations about surreal numbers. The whole chapter is also an original contribution. We study some continuity aspects to be able to study the derivation over surreal number with this fact in mind so that we may be able to reduce to solution to some polynomial differential equations to the Intermediate Value Theorem. We expect this consideration to lead to a better understanding of the behavior of the derivative over surreal numbers.

- Chapter 7 is our suggestion to handle oscillatory behaviors with surreal numbers. Extending them, we build the field of oscillating numbers that take into account the functions $\cos$ and $\sin$ that are known to have no satisfying extensions to surreal numbers. Thus, this number should be able to handle much more asymptotic behavior than surreal numbers alone and we expect them to encapsulate behaviors of continuous models of computations.
- Chapter 8 recaps what we presented in this thesis an gives some conjectures and perspective for further works.

Our main contributions can be divided into three parts which are the points of Chapters 5, 6 and 7 respectively.

- Our first contribution is to exhibit suitable fields to work on. For the sake of computability, we expect this field to be small enough to handle only computable ordinals. We also want them to be close under operations such as $\exp$, $\ln$, derivation and anti-derivation. This is a major change compared to previous work since previous attempts have considered at least uncountable ordinals[5] (see for instance [23, 24]). As we said, we do not just provide a construction that relies on non-computable but countable ordinals: we provide even smaller fields containing only computable ordinals. In particular, we show that ordinals up to ε_ω already form fields stable enough. Note that this ordinal is huge. However, it is stupidly small compared to the first uncountable ordinal, ω_1 or even compared to the first non-computable ordinal, ω_1^{CK} which is still countable.

  For this study, the main theorems developed in this thesis are unarguably the following:

  ➢ Theorem 5.1.6, which provides an example of field of surreal numbers which is both quite simple and still stable under exponential and logarithm. It is a consequence of Proposition 5.1.7, which establish a necessary and sufficient condition for a subfield of surreal numbers built from formal series to be stable under $\exp$ and $\ln$. This is the first step to get a field closed under even more operations.

  ➢ Theorems 5.1.10 and 5.1.11. These theorems decompose fields of surreal numbers that have a length[6] bounded by some ordinal λ as a strict hierarchy for fields of the form given by Theorem 5.1.6.

  ➢ Theorem 5.3.1 which extends the stability to the derivative and the anti-derivative over surreal numbers.

- We provide some notion of topology over the surreal numbers to handle the gaps which are almost everywhere in the surreal line. We develop and characterize notions of gap-connectedness, gap-compactness and gap-continuity for the functions over the surreals.

  In this domain, the main result we prove are:

  ➢ Proposition 6.2.10 characterizes gap-compact sets as closed, bounded and gap-connected sets.

  ➢ Proposition 6.2.16 gives a characterization of gap-compact sets that is analog to Borel's definition of compact sets.

  ➢ Theorem 6.3.8 is the Intermediate Value Theorem in the context of surreal numbers. It requires a stronger notion of continuity than the expected one.

- The whole chapter 7 is about providing a construction of a new field of numbers, oscillating numbers. These numbers solve the problem of defining $\cos \omega$ by considering it as a brand new number, together with many others. However, the construction remains quite simple.

[5] Uncountable ordinals are in particular not computable.
[6] The length of the signs sequence

Contents

Chapter 2

Well-ordered sets

Our work strongly relies on ordered sets and, more precisely, well-ordered sets. These concepts were originally introduced by Cantor ([17], with english translation [25]). Well-ordered sets are very natural when we think about orderings. Namely, given an element which is not maximal, it is always possible to say which one is the next element according to the ordering and, which is the most important, it is impossible to build an infinite decreasing sequence. This fact enables us to use induction over well ordered sets, which is very convenient. Finally, we stress the fact that it is possible to rank well ordered sets themselves with what is called their order types. The order type of a well ordered set A is just a specific well ordered set, an ordinal, which is order-isomorphic to A. They are much more easier to deal with as we have a clear construction for them.
In this state of the art chapter, we introduce theses concepts using already known works and also provide some proofs of classical results (namely Propositions 2.3.6, 2.4.2 and 2.4.3) but whose proofs[1] are hard to find in the literature.

- In Section 2.1, we quickly introduce some key notions about ordered sets.
- In Section 2.2 we define the ordinals from a Set Theoretic point of view. We also link them with arbitrary well ordered set.
- In Section 2.3, we dive deeper into the variety of operations ordinal numbers can support and take a look at special ordinals that have "stability" properties.
- Section 2.4 contains some bounds on the ordinals associated to specific constructions of well ordered sets.

2.1 Notions about orders

Let us got through some useful definition about ordered sets.

Definition 2.1.1 (Well ordered set)**.** Let $(A, \leq)$ be a (partially) ordered set. A is well-ordered (or well-founded) if every non-empty subset of A has minimal element.

In other words, A is well (partially) ordered if an only if it has no infinite decreasing sequence. Such sets are very useful for well-founded induction. The well founded induction works as follows: If we want to prove a property $P(x)$ over a well partially ordered set $(S, \leq)$,

1. We prove $P(x)$ for all minimal $x \in S$, *i.e* for all x such that $y \leq x \implies y = x$.
2. Assuming $P(y)$ for all y such that $y < x$, we prove $P(x)$

If these two properties are satisfied, then we proved that $P(x)$ holds for all $x \in S$. Indeed, if it was not true, the set $\{ x \in S \mid \neg P(x) \}$ (notice that this set is well defined in the sens of Set Theory) would not be empty and then have a minimal element by definition of a well (partially) ordered set. Call it x_0. Then either x_0 is minimal in S and therefore $P(x_0)$ holds by point 1, or, by minimality of x_0 we can deduce $P(x_0)$ by point 2. In both cases we reach a contradiction. Hence, the set was in fact empty.

Definition 2.1.2. Let $(S, \leq)$ a a (partially) ordered set and $A, B \subseteq S$. A and B are **cofinal** if for any $b \in B$, there is $a \in A$ such that $b \leq a$ and for any $a \in A$ there is $b \in B$ such that $a \leq b$. Similarly, A and B are **coinitial** if for any $b \in B$ there is $a \in A$ such that $a \leq b$ and for any $a \in A$ there is $b \in B$ such that $b \leq A$.

Example 2.1.3. For example, $\{0\}$ is coinitial with $\mathbb{N}$. Also, $\mathbb{Z}$ is both coinitial and cofinal with $\mathbb{R}$.

[1] At least proofs that does not reduce to say that it is a "well-known result".

2.2 Ordinal numbers

We first introduce ordinal numbers. They play a great role in the construction of surreal numbers we will see in Section 2.3. Moreover, ordinal numbers are a class of canonical well-ordered sets, namely, every totally well-ordered set is isomorphic (as an ordered set) to some ordinal.

Definition 2.2.1 (Ordinal numbers)**.** Let α be a set. α is an **ordinal number** if and only if the relation $\in$ is transitive over α and it defines a total strict well-order over α. In the context of ordinal numbers, we will possibly use $<$ instead of $\in$. We denote **Ord** for the class of all ordinal numbers.

Example 2.2.2. The empty set is an ordinal, as well as $\{\varnothing\}$. The later one is the unique ordinal with a single element. Indeed if $\{x\}$ was also an ordinal with $x \neq \varnothing$, there would be some $y \in x$. Therefore, by transitivity, $y \in \{x\}$. Then, $y = x$ hence $x \in x$ which is not possible since $\in$ is a total strict order.

Remark 2.2.3. The fact that $\in$ is a total strict order ensures the uniqueness of ordinal numbers. That way, we do not need the Axiom of Regularity to conclude to the unique definition of ordinal numbers. This uniqueness is stated in Proposition 2.2.5 Item (ii).

From the definition we immediately derive that

Proposition 2.2.4. *(i) If X is a set of ordinals such that $\in$ is transitive over X and such that for all $\alpha, \beta \in X$, $\alpha \in \beta$ or $\alpha = \beta$ or $\beta \in \alpha$, then, X is itself an ordinal.*

(ii) Let α be an ordinal number. Let $x \in \alpha$ and $\beta = \{y \in \alpha \mid y \in x\}$. Then β is an ordinal number. Moreover, if $\beta \neq \alpha$, then $\beta \in \alpha$.

(iii) If α is an ordinal, then $\alpha \cup \{\alpha\}$ is itself an ordinal number. It is called the successor of α .

Proposition 2.2.5. *(i) Let α, β be isomorphic ordinals as ordered sets. Then $\alpha = \beta$.*

(ii) Let α, β be ordinal numbers. Then either $\alpha \in \beta$ or $\beta \in \alpha$ or $\alpha = \beta$.

Corollary 2.2.6. *Let A be a set of ordinals. Then $\bigcup_{\alpha \in A} \alpha$ is an ordinal. It is denoted $\sup A$.*

If A has a greatest element, then it is $\sup A$. If not, $\sup A$ is the least ordinal containing A. The supremum is the way to build more and more ordinals. In fact, ordinal numbers can be divided into two categories:

Definition 2.2.7.
1. If an ordinal has the form $\alpha \cup \{\alpha\}$, it is called a **successor ordinal** (the successor of α) .
2. An ordinal which is not successor is called a **limit ordinal**. We denote ω for the least limit ordinal and **Lim** for the class of the limit ordinals.

Remark 2.2.8. If $\alpha \in \mathbf{Lim}$ then we have $\alpha = \sup \alpha$. Indeed, if $\beta \in \alpha$, then since $\alpha \in \mathbf{Lim}$ the successor of β must also be in α. Therefore, $\beta \in \sup \alpha$ as an element of $\beta \cup \{\beta\} \in \alpha$. Conversely, if $\beta \in \sup \alpha$, transitivity gives $\beta \in \alpha$.

In Proposition 2.2.5, we saw that ordinals are well defined and have a straightforward definition. Therefore, they may pretend to be standard representative for well-ordered set. More precisely, we now state that, in a sense, they are the only well total orders that exist; every well ordered set is order-isomorphic to some ordinal.

Theorem 2.2.9. *Let $(X, <)$ be a well total order. There is a unique ordinal which is order-isomorphic to $(X, <)$.*

Definition 2.2.10 (Order type)**.** We call the ordinal which isomorphic to a well ordered set, its **order type**.

Definition 2.2.11 (Cardinal)**.** A **cardinal** κ is an ordinal such that for all $\alpha \in \kappa$, there is no bijection between α and κ. A cardinal is **regular** if for any morphism φ between $\alpha \in \kappa$ and κ, φ is bounded above, *i.e* there is some $\beta \in \kappa$ such that for all $\gamma \in \alpha$, $\varphi(\gamma) \in \beta$.

Example 2.2.12. ω is regular, as well as ω_1 the first uncountable ordinal.

2.3 Operations over ordinal numbers, Cantor normal form

When counting or ranking things, we naturally end up with a well ordered set. We have a first element, then a second, and so on. In every day life, we of course use finite sets but it is still possible to extend the principle to infinity. Assume that we have ranked an infinite set of things with ranks from 0 to infinity, if we want to add a new element to our collection, it will be ranked after any natural number, it will be ranked ω. With that in mind we now can imagine that we want to rank the union of two ranked collections, one after the other. Here comes an idea of addition of well ordered set and therefore an addition of ordinal number.

2.3.1 Usual operation over ordinal numbers

Addition of well ordered sets can be seen a concatenation. Namely, if we have α element and then, after them, β elements, then we end up with $\alpha \oplus \beta$ elements.

Definition 2.3.1 (Ordinal addition)**.** (i) We define 0 to be $\varnothing$.

(ii) Denote $\alpha \oplus 0 = \alpha$.

(iii) Let α be an ordinal. Define $\alpha \oplus 1 = \alpha \cup \{\alpha\}$.

(iv) We define **natural number** as the successors of $0 : n = 0 \underbrace{\oplus 1 \oplus \cdots \oplus 1}_{n \text{ times}}$.

(v) Let α be an ordinal and $\beta = \gamma \oplus 1$ be a successor ordinal then we define $\alpha \oplus \beta = (\alpha \oplus \gamma) \oplus 1$

(vi) Let α be an ordinal and $\beta \in \mathbf{Lim}$, then we set $\alpha \oplus \beta = \sup \{ \alpha \oplus \gamma \mid \gamma \in \beta \}$.

Remark 2.3.2. Notice that this definition uses transfinite induction.

To make clearer the intuition about concatenation, we state the following lemma:

Lemma 2.3.3 (Concatenation, folklore)**.** *Let A and B be two well-ordered set with order type α and β respectively. Then, the concatenation of A and B, $(\{0\} \times A) \cup (\{1\} \times B)$, is well-ordered by the lexicographic order with order type $\alpha \oplus \beta$.*

The proof is straightforward by induction on β.

Definition 2.3.4 (Ordinal multiplication)**.** By transfinite induction, we define

(i) $\alpha \otimes 0 = 0$

(ii) $\alpha \otimes (\gamma \oplus 1) = (\alpha \otimes \gamma) \oplus \alpha$

(iii) For $\beta \in \mathbf{Lim}$, $\alpha \otimes \beta = \sup \{ \alpha \otimes \gamma \mid \gamma \in \beta \}$.

Remark 2.3.5. Even if $\oplus$ and $\otimes$ behave well over natural numbers, in the sense that they match the usual operations over natural numbers, even if both $\oplus$ and $\otimes$ are associative operators, none of them is commutative. That is why we do not use the classical operators $+$ and $\times$ that we will keep available to define commutative operations over ordinal numbers.

Ordinal multiplication is in fact the natural order type for the lexicographic order over a Cartesian product.

Proposition 2.3.6 (Cartesian product, folklore)**.** *Let A and B be two well-ordered sets with order type α and β respectively. Then, the Cartesian product $A \times B$ is well-ordered by the lexicographic order with order type $\beta \otimes \alpha$.*

Proof. We do it by induction on α.

- If $\alpha = 0$ then $A \times B = \varnothing$ which is well-ordered with order type $0 = \beta \otimes 0$.
- Assume the property for all well-ordered set A' with order type α. Let A of order-type $\alpha \oplus 1$, a its largest element and $A' = A \setminus \{a\}$. The order-type of A' is α. Then $A' \otimes B$ is well-ordered with order-type $\beta \otimes \alpha$. Since for any $b \in B$, (a, b) is larger than any element of $A' \times B$, the order type of $A \times B$ is indeed $\beta \otimes \alpha \oplus \beta = \beta \otimes (\alpha \oplus 1)$.
- Let α a limit ordinal and A of order type α. Let A_γ the initial segment of A of length γ with $\gamma < \alpha$. Then
$$A \times B = \bigcup_{\gamma < \alpha} A_\gamma \times B$$
is an increasing union of initial segments. By induction hypothesis, $A_\gamma \times B$ has order type $\beta \otimes \gamma$. Therefore $A \times B$ has order type $\sup_{\gamma < \alpha} \beta \otimes \gamma = \beta \otimes \alpha$.

□

Definition 2.3.7 (Ordinal exponentiation)**.** By transfinite induction, we define

(i) $\alpha^0 = 1$

(ii) $\alpha^{\gamma \oplus 1} = \alpha^\gamma \otimes \alpha$

(iii) For $\beta \in \mathbf{Lim}$, $\alpha^\beta = \sup \{ \alpha^\gamma \mid \gamma \in \beta \}$.

Example 2.3.8. We have for instance, $2^\omega = \omega$, $\omega^\omega = \sup \{ \omega^n \mid n \in \omega \}$ and $(\omega \oplus 42)^{\omega \oplus 57} = \omega^{\omega \oplus 57} \oplus \omega^{\omega \oplus 56} \otimes 42$.

Remark 2.3.9. For the sake of simplicity, we introduce priorities for ordinal operations to be the same as usual addition, multiplication and exponentiation: ordinal multiplication has priority over ordinal addition and ordinal exponentiation has priority over both of them. For instance

$$\alpha \oplus \beta \otimes \gamma = \alpha \oplus (\beta \otimes \gamma) \qquad \text{and} \qquad \alpha \oplus \beta^\gamma \otimes \delta = \alpha \oplus ((\beta^\gamma) \otimes \delta)$$

2.3.2 Cantor normal form and natural operations

We are now ready to give a major theorem about ordinal numbers. The following theorem is a generalization of Cantor's normal form theorem ([17]).

Theorem 2.3.10 (Cantor, [17])**.** *Let $\beta \geq 2$ and α be ordinal numbers. Then α can be written in a unique way as*

$$\beta^{e_1}k_1 \oplus \beta^{e_2}k_2 \oplus \cdots \oplus \beta^{e_n}k_n$$

with $0 \leq k_i < \beta$ and $e_1 > e_2 > \cdots > e_n > 0$ being ordinal numbers.
We call such an expression the normal form in base β of the ordinal α. The special case $\beta = \omega$ is called the ***Cantor's normal form****.*

An ordinal α in Cantor's normal form looks like a "polynomial" in ω excepts that the exponents may be any ordinal numbers and not only natural numbers. Such an expression may suggest a definition of an addition and a multiplication which behaves nicely with Cantor's normal form. They are called natural operations, or Hessenberg operations ([28]). The idea is to perform formal addition and multiplication of "polynomials".

Definition 2.3.11 (Natural addition)**.** Let $\alpha = \sum_{i=1}^{n}\omega^{\alpha_i}k_i$ and $\beta = \sum_{j=1}^{m}\omega^{\beta_j}l_j$ be two ordinal numbers in Cantor's normal form. Let p the number of element in the finite set $\{\alpha_i\}_i \cup \{\beta_j\}_j$. Let $\gamma_1 > \cdots > \gamma_p$ a decreasing enumeration of this set. Up to set $k_i = 0$ or $l_j = 0$, we assume, without loss of generality $n = m = p$ and for all $i \leq p$ $\alpha_i = \beta_i = \gamma_i$. We then define

$$\alpha + \beta = \sum_{i=1}^{p}\omega^{\gamma_i}(k_i + l_i)$$

where the sum is an ordinal sum.

Remark 2.3.12. It is the same definition as the addition of polynomial except that the exponents may not be natural numbers.

Remark 2.3.13. Notice that the ordinal sum behind the symbol Σ corresponds to the Hessenberg sum for the special case of "monomials" with decreasing exponents.

Remark 2.3.14. This addition behaves much more like the sum we would think at first thought. In particular, it defines a commutative monoid over ordinals numbers

Example 2.3.15. We give an example to compare both of the additions, $+$ and $\oplus$:

$$(\omega^2 2 + \omega 7 + 8) + (\omega^3 + 7) = \omega^3 + \omega^2 2 + \omega 7 + 15$$

whereas

$$(\omega^2 2 + \omega 7 + 8) \oplus (\omega^3 + 7) = \omega^3 + 7$$

and

$$(\omega^3 + 7) \oplus (\omega^2 2 + \omega 7 + 8) = \omega^3 + \omega^2 2 + \omega 7 + 8$$

Definition 2.3.16 (Natural multiplication)**.** Let $\alpha = \sum_{i=1}^{n}\omega^{\alpha_i}k_i$ and $\beta = \sum_{j=1}^{m}\omega^{\beta_j}l_j$ be two ordinal in Cantor's normal form. We define similarly:

$$\alpha \times \beta = \sum_{i,j}\omega^{\alpha_i + \beta_j}k_i l_j$$

2.3.3 Special classes of ordinal number

We already introduced the class **Lim** of limit ordinal numbers. This class is the class of ordinal numbers that are stable under the successor operation. Namely,

$$\alpha \in \mathbf{Lim} \iff \forall \beta \in \alpha \qquad \beta \cup \{\beta\} \in \alpha$$

in other words

$$\alpha \in \mathbf{Lim} \iff \forall \beta < \alpha \qquad \beta + 1 < \alpha$$

We have now seen other operations. We can also define classes of ordinal numbers that are close under addition, of multiplication or even exponentiation.

Lemma 2.3.17 (Additive ordinals, folklore)**.** *Let α be an ordinal number. We have $\beta + \gamma < \alpha$ (resp. $\beta \oplus \gamma < \alpha$) for all $\beta, \gamma < \alpha$ if and only if there is some ordinal number α' such that $\alpha = \omega^{\alpha'}$.*

Proof. (NC ⇒) Assume that for all $\beta, \gamma < \alpha$ we have $\beta + \gamma < \alpha$ (resp. $\beta \oplus \gamma < \alpha$). Let $\alpha = \sum_{k=1}^{n}\omega^{\alpha_i}n_i$ in Cantor's normal form. Let $\beta = \omega^{\alpha_1}$ and $\gamma = (n_1 - 1)\omega^{\alpha_1} + \sum_{k=1}^{n}\omega^{\alpha_i}$. If $\beta \neq \alpha$ then $\beta, \gamma < \alpha$ but still $\beta + \gamma = \alpha$ (resp. $\beta \oplus \gamma < \alpha$) which is a contradiction. Therefore $\beta = \alpha = \omega^{\alpha_1}$.

SC $\Leftarrow$ Assume now that $\alpha = \omega^{\alpha'}$ for some α'. Let $\beta = \sum_{k=1}^{n} \omega^{\beta_i} n_i < \alpha$ and $\gamma = \sum_{k=1}^{p} \omega^{\gamma_i} m_i < \alpha$ in Cantor's normal form. Then $\beta_1, \gamma_1 < \alpha'$. Then the first term of the Cantor's normal form of $\beta + \gamma$ (resp. $\beta \oplus \gamma$) is either $\omega^{\beta_1} n_1$, or $\omega^{\gamma_1} m_1$, or $\omega^{\beta_1}(n_1 + m_1)$. In all cases, $\beta + \gamma < \alpha$ (resp. $\beta \oplus \gamma < \alpha$).

□

Lemma 2.3.18 (Multiplicative ordinals, folklore)**.** *Let α be an ordinal number. We have $\beta\gamma < \alpha$ (resp. $\beta \otimes \gamma < \alpha$) for all $\beta, \gamma < \alpha$ if and only if there is some ordinal number α' such that $\alpha = \omega^{\omega^{\alpha'}}$.*

Proof. NC $\Rightarrow$ Assume that for all $\beta, \gamma < \alpha$ we have $\beta\gamma < \alpha$ (resp. $\beta \otimes \gamma < \alpha$). First, if there are $\beta, \gamma < \alpha$ such that $\beta + \gamma \geq \alpha$ (resp. $\beta \oplus \gamma \geq \alpha$), then either $\beta\beta \geq \alpha$ (resp. $\beta \otimes \beta \geq \alpha$) or $\gamma\gamma \geq \alpha$ (resp. $\gamma \otimes \gamma \geq \alpha$), depending on which of β or γ is the greatest. Then, from Lemma 2.3.17, $\alpha = \omega^{\alpha'}$ for some α'. Let $\alpha' = \sum_{k=1}^{n} \omega^{\alpha_i} n_i$ in Cantor's normal form. Let $\beta = \omega^{\omega^{\alpha_1}}$ and $\gamma = \omega^{(n_1-1)\omega^{\alpha_1} + \sum_{k=1}^{n} \omega^{\alpha_i}}$. If $\omega^\beta \neq \alpha$ then $\omega^\beta, \omega^\gamma < \alpha$ but still $\beta\gamma = \alpha$ (resp. $\beta \otimes \gamma < \alpha$) which is a contradiction. Therefore $\omega^\beta = \alpha = \omega^{\omega^{\alpha_1}}$.

SC $\Leftarrow$ Assume now that $\alpha = \omega^{\omega^{\alpha'}}$ for some α'. Let $\beta = \sum_{k=1}^{n} \omega^{\beta_i} n_i < \alpha$ and $\gamma = \sum_{k=1}^{p} \omega^{\gamma_i} m_i < \alpha$ in Cantor's normal form. Then $\beta_1, \gamma_1 < \omega^{\alpha'}$. Then the first term of the Cantor's normal form of $\beta\gamma$ (resp. $\beta \otimes \gamma$) is $\omega^{\beta_1 + \gamma_1} n_1 m_1$ (resp. $\omega^{\beta_1 \oplus \gamma_1}$). Hence, $\beta\gamma < \alpha$ (resp. $\beta \otimes \gamma < \alpha$).

□

Lemma 2.3.19 (ε-numbers, folklore)**.** *Let $\alpha > \omega$ be an ordinal number. We have $\beta^\gamma < \alpha$ for all $\beta, \gamma < \alpha$ if and only if $\alpha = \omega^\alpha$.*

Proof. NC $\Rightarrow$ Assume that for all $\beta, \gamma < \alpha$ we have $\beta^\gamma < \alpha$. First, let $\beta, \gamma < \alpha$ and denote $\delta = \max(\beta, \gamma, 2)$. Then $\delta^\delta \geq \beta \otimes \gamma$. By assumption we get $\beta \otimes \gamma < \alpha$ for all $\beta, \gamma < \alpha$. Applying Lemma 2.3.18, there is some α' such that $\alpha = \omega^{\omega^{\alpha'}}$. Assume $\omega^{\alpha'} \neq \alpha$. Then $\omega^{\alpha'} < \alpha$. Since $\omega < \alpha$ we have $\omega^{\omega^{\alpha'}} < \alpha$ by assumption, which is an immediate contradiction. Therefore, $\alpha = \omega^\alpha$.

SC $\Leftarrow$ Assume that $\alpha = \omega^\alpha$. Let $\beta, \gamma < \alpha$. Let $\omega^{\beta'} n$ be the first term of the Cantor's normal form of β, then the first term of the Cantor's normal form of β^γ is $\omega^{\beta' \otimes \gamma} m$ where m is either 1 or n. Since $\beta' \leq \beta$ we have also that $\beta' < \alpha$. Notice that we can also write $\alpha = \omega^{\omega^\alpha}$ and then Lemma 2.3.18 ensures that $\beta' \otimes \gamma < \alpha$. Finally, we get that $\beta^\gamma < \alpha$.

□

We give names to ordinal numbers satisfying the previous lemmas.

Definition 2.3.20. Let α be a ordinal number. α is

- an **additive ordinal** if there is some α' such that $\alpha = \omega^{\alpha'}$,
- a **multiplicative ordinal** if there is some α' such that $\alpha = \omega^{\omega^{\alpha'}}$,
- an ε-**number** if $\alpha = \omega^\alpha$.

Additive ordinals are also called additively undecomposable ordinal numbers. Indeed Lemma 2.3.17 says that such an ordinal cannot be written as the sum of two smaller ordinals. The same happen for multiplicative ordinals that are in fact multiplicatively undecomposable ordinal numbers.

2.4 Order types of composed well-ordered sets

When we have some well-ordered sets, say A and B, that we may assume to be contained in an other well-ordered set, C, we may wonder what is the order type of some composed sets such as $A \cup B$ or $A + B$ (if C supports an addition operation) or $\langle A \rangle$ the monoid generated by A (if C is a monoid). This section is about bounding the order types of these composed sets. The following results are known since years in the folklore but we provide proofs anyway, as we did not find a reference for them.

2.4.1 Union of well-ordered sets

Lemma 2.4.1 (Folklore)**.** *Let Γ be a totally ordered set, $A \subseteq \Gamma$ be a well-ordered subset with order type α. Let $g \in \Gamma$. Then the set $A \cup \{g\}$ is well ordered with order type at most $\alpha + 1$.*

Proof. We prove it by induction on α.

- If $\alpha = 0$ then $A \cup \{g\}$ has only one element, and then has order type $1 = \alpha + 1$.
- If $\alpha = \gamma + 1$ is a successor ordinal. Let u the largest element in A. If $u \leq g$ then $A \cup \{g\}$ has indeed order type at most $\alpha + 1$. If not, then, by induction hypothesis, $(A \setminus \{u\}) \cup \{g\}$ has order type at most $\gamma + 1 = \alpha$. Then $A \cup \{g\} = ((A \setminus \{u\}) \cup \{g\}) \cup \{u\}$ has order type at most $\alpha + 1$.
- If α is a limit ordinal. If g is larger than any element of A, then $A \cup \{g\}$ has order type $\alpha + 1$. If not, let $a_0 \in A$ such that $a_0 \geq g$. For $a \in A$ such that $a > a_0$ set

  $$B_a = \{g\} \cup \{\, a' \in A \mid a' < a \}$$

  Since α is limit, we have

  $$A \cup \{g\} = \bigcup_{a > a_0} B_a$$

  and each of the element in the union is an initial segment of $A \cup \{g\}$. We also denote α_a the order type of the set $\{\, a' \in A \mid a' < a \}$. In particular, $\alpha_a < \alpha$. Using induction hypothesis, B_a has order type at most $\alpha_a + 1$. Then, since we have an increasing union of initial segments, the order type of $A \cup \{g\}$ is at most

  $$\sup \{\, \alpha_a + 1 \mid a > a_0 \} = \sup \{\, \alpha' + 1 \mid \alpha' < \alpha \} = \alpha$$

  since α is a limit ordinal.

We conclude thanks to the induction principle. □

Proposition 2.4.2 (Union of well-ordered sets, folklore)**.** *Let Γ be a totally ordered set $A, B \subseteq \Gamma$ be non-empty well-ordered subsets with respective order types α and β. Then the subset $A \cup B$ is well ordered with order type at most $\alpha + \beta$.*

Proof. $A \cup B$ is well-ordered. Indeed, if we have an infinite decreasing sequence of $A \cup B$, then we can extract either an infinite one for either A or B which is not possible. It remains to show the bound on its order type. We do it by induction over α and β.

- If $\alpha = \beta = 1$, then $A \cup B$ has at most two elements. Then, its order type is at most $2 = \alpha + \beta$.
- If α or β is a successor ordinal. Since both cases are symmetric, we assume without loss of generality that $\beta = \gamma + 1$. Let u be the largest element of B and $C = B \setminus \{u\}$. Then, by induction hypothesis, $A \cup C$ has order type at most $\alpha + \gamma$. Using Lemma 2.4.1, we get that the order type of $A \cup B$ is at most $\alpha + \gamma + 1 = \alpha + \beta$.
- If α and β are limit ordinal. A or B must be cofinal with $A \cup B$. For instance say it is A. For $a \in A$, let

  $$A_a = \{\, a' \in A \mid a' < a \} \qquad \text{and} \qquad B_a = \{\, b \in B \mid b < a \}$$

  We have

  $$A \cup B = \bigcup_{a \in A} A_a \cup B_a$$

  Since A is cofinal with $A \cup B$, it is an increasing union of initial segments. Let α_a be the order type of A_a and β_a the one of B_a. We have $\alpha_a < \alpha$ and $\beta_a \leq \beta$. By induction hypothesis, $A_a \cup B_a$ has order type at most $\alpha_a + \beta_a$. Then $A \cup B$ has order type at most

  $$\sup \{\, \alpha_a + \beta_a \mid a \in A \} \leq \alpha + \beta$$

We conclude the proof using the induction principle. □

2.4.2 Sum of well-ordered subsets of a monoid

Assume now that we have well-ordered subsets of an ordered Abelian additive monoid. A very natural subset to build is the set of elements of the group that are a sum of one element of the first set and one element of the second one. In the worst case, we build something that looks like a Cartesian product. Then we can expect the natural product of ordinal numbers to play a role in the sum of well-ordered subsets, and it actually does.

Proposition 2.4.3 (Folklore)**.** *Let Γ be an ordered Abelian additive monoid and $A, B \subseteq \Gamma$ be non-empty well-ordered subsets with respective order types α and β. Then the subset $A + B = \{\, a + b \mid a \in A \quad b \in B \}$ is well ordered with order type at most $\alpha\beta$.*

Proof. We prove it by induction over α and β.

- If $\alpha = \beta = 1$, then $A + B$ has only one element, then has order type $1 = \alpha\beta$.
- If α or β is not an additive ordinal. Let say $\beta = \gamma + \delta$ with $\gamma, \delta < \beta$. We choose γ, δ such that $\gamma + \delta = \gamma \oplus \delta$. Let B_1 the initial segment of length γ of B. Let $B_2 = B \setminus B_1$. B_2 has order type δ. Then, by induction hypothesis, $A + B_1$ has order type at most $\alpha\gamma$ and $A + B_2$ has order type at most $\alpha\delta$. Then, using Proposition 2.4.2, $A + B$ has order type at most $\alpha\gamma + \alpha\delta = \alpha\beta$.
- If both α and β are additive ordinals. Assume $A + B$ has order type more than $\alpha\beta$. Let $a + b \in A + B$ such that $C := \{ c \in A + B \mid c < a + b \}$ has order type $\alpha\beta$. Let

  $$A_0 = \{ a' \in A \mid a' < a \} \text{ and } B_0 = \{ b' \in B \mid b' < b \}$$

  and α_0 and β_0 their respective order types. We have

  $$C \subseteq (A_0 + B) \cup (A + B_0)$$

  Using induction hypothesis and Proposition 2.4.2, C has order type at most $\alpha_0\beta + \alpha\beta_0$. Since $\alpha_0 < \alpha$ and $\beta_0 < \beta$, we have $\alpha_0\beta < \alpha\beta$ and $\alpha\beta_0 < \alpha\beta$. α and β being additive ordinal, $\alpha\beta$ is itself an additive ordinal and then C has order type less than $\alpha\beta$, what is a contradiction. Then $A + B$ has order type at most $\alpha\beta$.

We conclude thanks to the induction principle. □

Warning 2.4.4. In the previous proof, we can conclude that $\alpha_0\beta < \alpha\beta$ because we are using the natural product. If we were using the $\otimes$ operator, this inequality would not necessarily hold.

2.4.3 Monoid generated by a well-ordered subset of a monoid

We can go further and wonder what would happen if we iterate the previous construction. Namely, if we look at a monoid generated by some well ordered subset of some Abelian group.

Proposition 2.4.5 ([53, Weiermann, Corollary 1])**.** *Let Γ be an ordered Abelian group and $S \subseteq \Gamma_+$ be a well-ordered subset with order type α. Then, $\langle S \rangle$, the monoid generated by S in Γ is itself well-ordered with order type at most $\omega^{\widehat{\alpha}}$ where, if the Cantor normal form of α is*

$$\alpha = \sum_{i=1}^{n} \omega^{\alpha_i} n_i$$

then

$$\widehat{\alpha} = \sum_{i=1}^{n} \omega^{\alpha_i'} n_i$$

and

$$\beta' = \begin{cases} \beta + 1 & \text{if } \beta \text{ is an } \varepsilon\text{-number} \\ \beta & \text{otherwise} \end{cases}$$

In particular, $\langle S \rangle$ has order type at most $\omega^{\omega\alpha}$ (commutative multiplication).

2.4.4 Order type of the set of finite sequences

Finally, we also have an explicit bound when we look at the finite sequences over a well-ordered set, which also form a well-ordered set for the corresponding lexicographic order (up to add a blank symbol as new minimum to compare the words).

Theorem 2.4.6 ([20, de Jongh, Parikh, Theorem 3.11] and [42, Schmidt, Theorem 2.9])**.** *Let $(X, \leq)$ be a well ordered set with order type α. Let X^* be the set of finite sequences over X. Let β the order type of X^*. We have*

$$\beta \leq \begin{cases} \omega^{\omega^{\alpha-1}} & \text{if } \alpha \text{ is finite} \\ \omega^{\omega^{\alpha+1}} & \text{if } \varepsilon \leq \alpha < \varepsilon + \omega \text{ for some } \varepsilon\text{-number } \varepsilon \\ \omega^{\omega^{\alpha}} & \text{otherwise} \end{cases}$$

Chapter 3

Surreal numbers

In 1976, Conway published his classical book [18] "*On numbers and Games*". Motivated by Games Theory, he introduced new numbers and came up with a construction of "all numbers great and small". This relies, among other things, on Sets Theory. Since his construction contains "all" the numbers (assuming that only numbers that support an ordering are considered), it contains in particular real numbers but also ordinal numbers. More precisely, real numbers and ordinal numbers can be identified as subsets of the class of surreal numbers, which are all the numbers that we can put an order onto. In this chapter we take a look at Conway's construction and Gonshor's formalism [26] of surreal numbers.
This chapter mostly presents what has already be done in the literature about surreal numbers. However, we still provide some new bounds about what is called the nested truncation rank in Lemma 3.8.24, Corollary 3.8.25, Propositions 3.9.29 and 3.9.31. These bounds will be used in Chapter 5.

- Section 3.1 introduces two different ways to build surreal numbers.
- Section 3.2 is dedicated to the operations we can use over surreal numbers.
- Section 3.3 gives a construction of a normal form for surreal numbers and relates this normal form to the initial construction of surreal numbers.
- Section 3.4 relates additive ordinals, multiplicative ordinals and ε-numbers to some identifiable structures inside the class of surreal numbers.
- Section 3.5 introduces the concept of substructure and parametrization of some specific class of surreal numbers
- Section 3.6 builds an exponential and a logarithm over surreal numbers.
- Section 3.7 defines a particular class of surreal number to be used as elementary bricks, or leafs of a tree representation of surreal numbers provided by the next section.
- Section 3.8 provides a tree representation and studies its properties.
- Sections 3.9 and 3.10 use the tree representation to define a derivation of surreal number and a way to get an anti-derivative of a given surreal number.

3.1 Construction of surreal numbers

3.1.1 Conway's construction

Definition 3.1.1 (Conway's construction of surreal numbers and their order [18])**.** We define surreal numbers by transfinite induction as follows:

- Start with "nothing", that is to say consider the pair $[\varnothing \mid \varnothing]$. Call it 0 and set $0 \leq_0 0$. Set $\mathcal{N}_0 = \{0\}$.
- Assume we have built a set of numbers $\mathcal{N}_\alpha$ for some ordinal number α and a preorder on it, $\leq_\alpha$. Let $L, R \subseteq \mathcal{N}_\alpha$ such that
$$\forall \ell \in L \quad \forall r \in R \qquad \ell <_\alpha r$$
Then we say that the pair $[L \mid R]$ is a new number and define
$$\mathcal{N}_{\alpha+1} = \mathcal{N}_\alpha \cup \{\, [L \mid R] \mid L, R \subseteq \mathcal{N}_\alpha \}$$

We now define the preorder. For $x = [L \mid R] \in \mathcal{N}_{\alpha+1}$ and $y = [L' \mid R'] \in \mathcal{N}_{\alpha+1}$ we define by well-founded induction on x and y that:

$$x \leq_{\alpha+1} y \qquad \textit{iff} \qquad \begin{cases} x \leq_\alpha y & \text{if } x, y \in \mathcal{N}_\alpha \\ x < R' \wedge L < y & \text{otherwise} \end{cases}$$

Notice that at any point in the induction, sets L, R, L', R' may be empty. Therefore, the corresponding inequalities become trivial. Another point is that the preorder is well defined since no element of L, R, L' or R' is in $\mathcal{N}_{\alpha+1} \backslash \mathcal{N}_\alpha$. In other words, there is a pair of ordinal indices that is lexicographically decreasing.

- Assume we have built $\mathcal{N}_\beta$ for $\beta < \alpha$ and α a limit ordinal. We define

$$\mathcal{N}_\alpha = \bigcup_{\beta < \alpha} \mathcal{N}_\beta$$

and $$x \leq_\alpha y \qquad \Longleftrightarrow \qquad \exists \beta < \alpha \quad x \leq_\beta y$$

Notice that by induction on α, each relation $\leq_\alpha$ is indeed a preorder. Then we can define

$$\mathcal{N} = \bigcup_{\alpha \in \mathbf{Ord}} \mathcal{N}_\alpha$$

$$\forall x, y \in \mathcal{N} \qquad x \leq y \Longleftrightarrow \exists \alpha \in \mathbf{Ord} \quad x \leq_\alpha y$$

and $\leq$ is indeed a preorder over $\mathcal{N}$. The class of **surreal numbers** in Conway's point of view is

$$\mathbf{No} = \mathcal{N} / =$$

where $=$ is the equivalence relation on $\mathcal{N}$ associated to the preorder $\leq$. In that way, $\leq$ defines a total order over **No**.

The previous definition is quite complicated and was not exactly stated like that in [18]. However, the original definition lacks formality and we just give here a more detailed definition.

Definition 3.1.2. The **birthday** of a surreal number $x \in \mathbf{No}$ is the minimum ordinal α such that there are sets $L, R \subseteq \mathcal{N}$ such that $[L \mid R] \in \mathcal{N}_\alpha$ and $x = [L \mid R]$.

The only number whose birthday is 0 is called the surreal number 0. There are only two numbers whose birthday is 1. Call them -1 and 1. All these names are consistent with the definitions of operations we will give in the following (see Section 3.2). We have

$$-1 = [\varnothing \mid \{0\}] \qquad \text{and} \qquad 1 = [\{0\} \mid \varnothing]$$

On day 2, we have four numbers which are

$$-2 = [\varnothing \mid \{0, -1\}] \qquad -\frac{1}{2} = [\{-1\} \mid \{0\}] \qquad \frac{1}{2} = [\{0\} \mid \{1\}] \qquad 2 = [\{0, 1\} \mid \varnothing]$$

Notice that we could have considered the pair $[\{-1\} \mid \{1\}]$ but looking at the definition of $=$, we can see that

$$[\{-1\} \mid \{1\}] = [\varnothing \mid \varnothing] = 0$$

If we keep this process going on, we will define all the dyadic numbers. Actually the dyadic numbers can be identified with exactly all surreal numbers whose birthday is finite. Using this encoding of the dyadic numbers into the surreal numbers, we can now embed all the other real numbers. Indeed, consider $x \in \mathbb{R}$ not dyadic. Then x can be identified with

$$\left[\left\{ \frac{m}{2^n} < x \;\middle|\; m \in \mathbb{Z},\ n \in \mathbb{N} \right\} \;\middle|\; \left\{ \frac{m}{2^n} > x \;\middle|\; m \in \mathbb{Z},\ n \in \mathbb{N} \right\} \right]$$

We can also see that every natural number can be embedded as $n = [\llbracket 0\ ;\ n-1 \rrbracket \mid \varnothing]$. Following the same idea, ordinal numbers can be seen inductively as $\alpha = [\{ \beta \in \mathbf{Ord} \mid \beta < \alpha\} \mid \varnothing]$.

Notation. We may forget the brackets $\{\}$ when considering $[L \mid R]$. For instance, $[0, 1 \mid 2]$ will stand for $[\{0, 1\} \mid \{2\}]$.

3.1.2 Gonshor's construction

In 1986, Gonshor published his classical book [26] and gave an alternative definition and presentation of the theory of surreal numbers. He mostly considers sequences of pluses and minuses that terminates at some ordinal step. Actually, this idea was first introduced by Conway ([18]) himself. Then main difference with Gonshor's point of view is that Conway needed to introduce at least the addition of surreal numbers to define the signs sequences. Gonshor took the opposite direction: he started with signs sequences. This idea is helpful since we then can define surreal numbers thanks to some already known concepts (sequences, lexicographic order, ...) and is not just a pure new formal definition. Following Gonshor, we then give the following definition:

Definition 3.1.3 (Gonshor's construction of surreal numbers)**.** The class **No** $= \{+,-\}^{<\mathbf{Ord}}$ is the class of all binary sequences of some ordinal length $\alpha \in$ **Ord** where **Ord** denotes the class of all ordinal numbers. In other words, **No** corresponds to functions of the form $x : \alpha \to \{+,-\}$ where the ordinal number α is the **length** of the surreal number x. The **length** α of the surreal number x is denoted by $|x|_{+-}$ (the idea of this notation is to "count" the number of pluses and minuses).
Similarly, we denote the (ordinal) number of pluses in the signs sequence of x by $|x|_+$.
For all surreal number x, denote x_+ the surreal number whose signs sequence is the one of x followed by a plus. More precisely, we have

$$x_+ : \begin{cases} |x|_{+-}+1 & \to & \{+,-\} \\ \alpha & \mapsto & \begin{cases} x(\alpha) & \text{if } \alpha < |x|_{+-} \\ + & \text{if } \alpha = |x|_{+-} \end{cases} \end{cases}$$

The order over surreal number is then given by the lexicographic order over the signs sequences. More precisely we say that $- < \square < +$ where $\square$ is a blank symbol and to compare x and y, we append blank symbols to the end of the shortest one and then compare lexicographically.

Note that **No** is not a set but a proper class, and all the relations and functions we shall define on **No** are going to be class-relations and class-functions, usually constructed by transfinite induction. We will also see that we can give **No** a structure of class-field.
In the following, surreal numbers will be handled without necessarily use an explicit formulation of the underlying sign sequence. For instance, we will use the number 10 to speak about the surreal number :

$$10 : \begin{cases} 10 = [\![0\ ;\ 9]\!] & \to & \{+,-\} \\ k & \mapsto & + \end{cases}$$

Therefore, when we want to handle the pluses and minuses, we will speak about the **signs sequence** of a surreal number.

Notation. We will use some notations to describe the signs sequences of surreal numbers. We will use the following grammar :

$$\begin{aligned} S &\to \varepsilon \mid (S)^\alpha \mid SS \mid A \\ A &\to B+ \mid B- \\ B &\to \varepsilon \mid A \end{aligned}$$

where α is an ordinal number and the rules can be applied an ordinal number of times. The expression gives the values of the signs sequence in order as expected. For instance, $(+-+-)$ is the surreal number x of length 4 such that $x(0) = x(2) = +$ and $x(1) = x(3) = -$. Similarly, $(+-)^\omega(+)$ is the surreal number y of length $\omega+1$ such that for all $k \in \mathbb{N}$, $y(2k) = +$ and $y(2k+1) = -$ and such that $y(\omega) = -$.
Another useful concept is the notion of prefixes. We take a notation inspired by the Python programming language. If x is a surreal number of length α and β and γ are two ordinal numbers such that $\beta \leq \gamma \leq \alpha$, we denote $x[\beta:\gamma]$ the surreal number defined as follows:

$$\forall \delta \in \mathbf{Ord} \qquad \beta \oplus \delta < \gamma \implies x[\beta:\gamma](\delta) = x(\beta \oplus \delta)$$

and if $\beta \oplus \delta \geq \gamma$, $x[\beta:\gamma](\delta)$ is undefined. We will also let for $\beta, \gamma \leq \alpha$,

$$x[\beta:] = x[\beta:\alpha] \qquad \text{and} \qquad x[:\gamma] = x[0:\gamma]$$

Finally, we introduce the set $\mathbf{No}_\alpha = \left\{ x \in \mathbf{No} \;\middle|\; |x|_{+-} < \alpha \right\}$ for all ordinal α. This is the set of surreal number whose lengths are bounded above by α.

Definition 3.1.4 (Prefix)**.** Let x and y be surreal numbers. y is said to be a **prefix** of x *iff* there is some ordinal $\alpha \leq |x|_{+-}$ such that $y = x[:\alpha]$. When this is true, we write $y \sqsubseteq x$. If $x \sqsubseteq y$ but $x \neq y$, we write $x \sqsubset y$. If so, we say that x is **simpler** than y.

We now can define the operation $[\cdot \mid \cdot]$ over surreal numbers. It relies on a fundamental existence theorem which follows:

Theorem 3.1.5 ([26, Gonshor, Theorem 2.1])**.** *Let L, R be two sets of surreal numbers such that $L < R$. There is a unique shortest surreal number x such that $L < x < R$. Moreover, x is a prefix of any y such that $L < y < R$.*

Definition 3.1.6 ([26, Gonshor])**.** We define $[L \mid R]$ to be the unique shortest surreal number x such that $L < x < R$. If $x = [L \mid R]$ we say that the writing $[L \mid R]$ is a **representation** of x. Fixing a representation $x = [L \mid R]$, we say that the element of L are **left elements** of x and that the elements of R are **right elements** of x.

This definition actually corresponds to the definition of $[\cdot \mid \cdot]$ given in section 3.1.1. Indeed, we can prove by induction that $\mathcal{N}_\alpha/=$ and $\left\{ x \in \mathbf{No} \,\middle|\, |x|_{+-} \leq \alpha \right\}$ are order-isomorphic and that all the isomorphisms are successive extensions of each other. In particular $[\cdot \mid \cdot]$ is the same in both points of view and therefore :

Lemma 3.1.7 ([26, Gonshor, Theorem 2.5])**.** *Let $x = [L \mid R]$ an $y = [S \mid T]$ be two surreal numbers. Then*

$$x \leq y \iff x < T \wedge L < d$$

Proof of the isomorphism.

- For $\alpha = 0$, $\mathcal{N}_0 = \{[\varnothing \mid \varnothing]\}$ which exactly corresponds to $\left\{ x \in \mathbf{No} \,\middle|\, |x|_{+-} \leq 0 \right\}$.
- Assume that $\mathcal{N}_\alpha/=$ and $\left\{ x \in \mathbf{No} \,\middle|\, |x|_{+-} \leq \alpha \right\}$ are order-isomorphic with some isomorphism Φ. Recall that

  $$\mathcal{N}_{\alpha+1} = \mathcal{N}_\alpha \cup \{ [L \mid R] \mid L, R \subseteq \mathcal{N}_\alpha \}$$

  By induction hypothesis, for $L, R \subset \mathcal{N}_\alpha$, there are $L', R' \subseteq \left\{ x \in \mathbf{No} \,\middle|\, |x|_{+-} \leq \alpha \right\}$ such that $\Phi(L/=) = L'$ and $\Phi(R/=) = R'$. We then define

  $$\Phi'(x) = \begin{cases} \Phi(x) & \text{if } x \in \mathcal{N}_\alpha/= \\ [L' \mid R'] & \text{if } x \notin \mathcal{N}_\alpha/= \text{ is the equivalence class of } [L \mid R] \text{ with } L, R \subseteq \mathcal{N}_\alpha \end{cases}$$

  By definition, this function is surjective. Let now $x, y \in \mathcal{N}_{\alpha+1}/=$ such that $x < y$. Since the definition of the order is the same in $\mathcal{N}_{\alpha+1}/=$ and in $\mathbf{No}$, we then get $\Phi'(x) < \Phi'(y)$. Finally, Φ' is an order isomorphism. Moreover, Φ is a restriction of Φ'.
- Assume that $\mathcal{N}_\beta/=$ and $\left\{ x \in \mathbf{No} \,\middle|\, |x|_{+-} \leq \beta \right\}$ are order-isomorphic with some isomorphism Φ_β for all $\beta < \alpha$ and that if $\gamma < \beta$, Φ_γ is a restriction of Φ_α. We define an order isomorphism Φ between $\mathcal{N}_{\alpha+1}/=$ and $\left\{ x \in \mathbf{No} \,\middle|\, |x|_{+-} \leq \alpha + 1 \right\}$ by $\Phi(x) = \Phi_{\beta(x)}(x)$ where $\beta(x)$ is the least ordinal such that $x \in \mathcal{N}_\beta/=$. The induction hypothesis ensures that Φ is itself an order isomorphism and that all the Φ_βs are restrictions of Φ.

□

3.1.3 Properties of $[\cdot \mid \cdot]$

We give here some natural properties of the operator $[\cdot \mid \cdot]$.
First, we can characterize the surreal numbers that have a representation of the form $[L \mid R]$ with either $L = \varnothing$ or $R = \varnothing$.

Lemma 3.1.8 ([26, Theorem 2.2])**.** *Let L, R be two sets of surreal numbers. If $R = \varnothing$ then $[L \mid R]$ consists solely of pluses. Similarly, if $L = \varnothing$, $[L \mid R]$ consists solely of minuses.*

Remark 3.1.9. Not that if $L = R = \varnothing$, the signs sequence of $[L \mid R]$ may consist on solely pluses and solely minuses. This means that the signs sequence if empty: It is 0.

Since the sequences are of ordinal length, such numbers consists of an ordinal number of pluses and an ordinal number of minuses. These kind of surreal numbers are naturally well-ordered and reverse-well-ordered respectively. Therefore, we can identify them with the ordinal numbers and opposites of ordinal numbers respectively. At least as ordered classes, this identification is trivial. However ordinal numbers come with some very nice operations such as the natural operations (see Definitions 2.3.11 and 2.3.16). We will see in Section 3.2 that theses operations match the field operations we can define on all surreal numbers.
Speaking about operations, since we have defined the sets $\mathbf{No}_\alpha$, it would be nice to be able to bound from above the length of surreal numbers we can build thanks to $\mathbf{No}_\alpha$. Of course $\mathbf{No}_\alpha$ will never be stable under $[\cdot \mid \cdot]$ since by definition $[\{ x \in \mathbf{No}_\alpha \mid x < a \} \mid \{ x \in \mathbf{No}_\alpha \mid x \geq a \}]$ is not an element of $\mathbf{No}_\alpha$. However, we can have an explicit upper bound.

Lemma 3.1.10 ([26, Gonshor, Theorem 2.2])**.** *Let L, R be two sets of surreal numbers. Let α the ordinal defined by $\alpha = \min \left\{ \beta \in \mathbf{Ord} \,\middle|\, \forall x \in L \cup R \quad |x|_{+-} < \beta \right\}$. Then $|[L \mid R]|_{+-} \leq \alpha$.*

The operator $[\cdot \mid \cdot]$ is surjective and we can give a very special representation of each surreal number.

Theorem 3.1.11 ([26, Gonshor, Theorem 2.8]). *For all surreal number x, we have*

$$x = [\{y \in \mathbf{No} \mid y \sqsubset x \wedge y < x\} \mid \{y \in \mathbf{No} \mid y \sqsubset x \wedge y > x\}]$$

The previous theorem gives a very compact representation of x. In particular, every left element and every right element is simpler than x.

Definition 3.1.12 ([26, Gonshor]). For x a surreal number, the writing

$$[\{y \in \mathbf{No} \mid y \sqsubset x \wedge y < x\} \mid \{y \in \mathbf{No} \mid y \sqsubset x \wedge y > x\}]$$

is called the **canonical representation** of x.

3.2 Field operations

Definition 3.2.1 (Ring operations). Ring operations $+, \cdot$ on **No** are defined by transfinite induction on simplicity as follows:

$$x + y := [x' + y, x + y' \mid x'' + y, x + y'']$$
$$-x = [-x'' \mid -x']$$
$$xy := [x'y + xy' - x'y', x''y + xy'' - x''y'' \mid x'y + xy'' - x'y'', x''y + xy' - x''y']$$

where x' (resp. y') ranges over the numbers simpler than x (resp. y) such that $x' < x$ (resp. $y' < y$) and x'' (resp. y'') ranges over the numbers simpler than x (resp. y) such that $x < x''$ (resp. $y < y''$); in other words, when $x = [x' \mid x'']$ and $y = [y' \mid y'']$ are the canonical representations of x and y respectively.

Remark 3.2.2. The expression for the product may seem not intuitive, but actually, it is basically inspired by the fact that we expect $(x - x')(y - y') > 0$, $(x - x'')(y - y'') > 0$, $(x - x')(y - y'') < 0$ and $(x - x'')(y - y') < 0$ whenever $x' < x < x''$ and $y' < y < y''$.

3.2.1 Addition over surreal numbers

Proposition 3.2.3 ([26, Gonshor, Theorem 3.1]). *For all surreal numbers x and y, $x + y$ is well defined and for all surreal numbers x, y and z,*

$$x + y = y + x \qquad \textit{and} \qquad y > z \implies y + x > z + x$$

The original proof uses an induction over the ordinal sum of the lengths.

Proof. Let α, β, γ be the lengths of x, y, z respectively. We proceed by induction over (α, β, γ) with the lexicographic order, $<_{lex}$, and use the following induction hypothesis:

$x + y$ and $x + z$ are well defined and $+$ is commutative. Moreover, if $y > z$ then $y + x > z + x$.

- If $\alpha = \beta = \gamma = 0$, then $x = y = z$ and $x + y = 0$ which is well defined and the implication holds since $y > z$ does not.
- Consider x, y, z of respective lengths α, β, γ and assume that for all u, v, t of respective lengths λ, μ, ν, such that $(\lambda, \nu, \mu) <_{lex} (\alpha, \beta, \gamma)$, the property holds. Note that $x', x'' \sqsubset x$ and $y', y'' \sqsubset y$. Therefore we can use the induction hypothesis on $(x', y, z), (x'', y, z), (x, y', z)$ and (x, y'', z). Then the sets

$$L = \left\{ x' + y, x + y' \,\middle|\, \begin{matrix} x' \sqsubset x & x' < x \\ y' \sqsubset y & y' < y \end{matrix} \right\} \quad \text{and} \quad R = \left\{ x'' + y, x + y'' \,\middle|\, \begin{matrix} x'' \sqsubset x & x < x'' \\ y'' \sqsubset y & y < y'' \end{matrix} \right\}$$

are well defined. We then just need to show that $L < R$. By induction hypothesis, we have

$$x' < x'' \implies x' + y < x'' + y \tag{3.1}$$
$$y < y'' \implies x' + y < x' + y'' \tag{3.2}$$
$$x' < x \implies x' + y'' < x + y'' \tag{3.3}$$
$$(3.2) \wedge (3.3) \implies x' + y < x + y'' \tag{3.4}$$
$$y' < y'' \implies x + y' < x + y'' \tag{3.5}$$
$$x < x'' \implies x + y' < x'' + y' \tag{3.6}$$
$$y' < y'' \implies x'' + y' < x'' + y \tag{3.7}$$
$$(3.6) \wedge (3.7) \implies x + y' < x'' + y \tag{3.8}$$

so that $L < R$ and $[L \mid R]$ is well defined. Moreover, the definition of $y + x$ gives exactly the same sets L and R. Therefore, $x + y = y + x$. The very same proof with x and z ensures that $x + z$ is well defined and $x + z = z + x$.

Assume now that $y > z$.

➢ If $z \sqsubset y$ then $x+z$ appears in the definition of $y+x$ as a left element. Therefore $y+x > z+x$.

➢ If $y \sqsubset z$ then $y+x$ appears in the definition of $z+x$ as a right element. Therefore $y+x > z+x$.

➢ Otherwise, let d the longest common prefix of y and z (i.e. $d = [z \mid y]$). Then $d \sqsubset y$, $d \sqsubset z$ and $z < d < y$. $d+x$ appears in the definition of $y+x$ as a left element, and in the definition of $z+x$ as a right element. Therefore $z+x < d+x < y+x$.

□

The addition operation is well-defined. However, the current definition is quite restrictive since it requires canonical representations for x and y but does not ensure to provide a canonical representation as a result. This problem can be solved by the fact that the addition is our first example of operations defined by the operator $[\cdot \mid \cdot]$ that satisfies the uniformity property.

Definition 3.2.4 (Uniformity property)**.** Let $n \in \mathbb{N}$ and F be a (possibly partial) function defined on the class of the $2n$-tuple of subsets of **No** $(L_1, \ldots, L_n, R_1, \ldots, R_n)$ such that $L_i < R_i$ for all $i \in [\![1 \,;\, n]\!]$, to the class of ordered pairs (L, R) of subsets of **No** such that $L < R$. We say that F defines a function $f : \mathbf{No}^n \to \mathbf{No}$ if an only if

$$(L, R) = F(L_1, \ldots, L_n, R_1, \ldots, R_n) \iff f([L_1 \mid R_1], \ldots, [L_n \mid R_n]) = [L \mid R]$$

whenever for all $i \in [\![1 \,;\, n]\!]$, $x_i = [L_i \mid R_i]$ is the canonical representation of x_i. The function F is said to have the **uniformity property** if the above equivalence if valid if we do not require a canonical representation anymore. By extension, we say that f has the uniformity property if there is some F that defines it and that has the uniformity property. When the context is clear, we shall not specify the function F and just say that f has the uniformity property.

Note that in the previous definition, nothing prevents the definition to be inductive. Note also that there may exist several functions F that define the same function f.

Example 3.2.5. For instance, the addition is defined by the following class-function:

$$F(L,S,R,T) = \left(\left\{ x' + [S \mid T], [L \mid R] + y' \;\middle|\; \begin{array}{c} x' \in L \\ y' \in S \end{array} \right\}, \left\{ x'' + [S \mid T], [L \mid R] + y'' \;\middle|\; \begin{array}{c} x'' \in R \\ y'' \in T \end{array} \right\} \right)$$

Namely, if $x = [L \mid R]$ and $y = [S \mid T]$ we indeed have $F(L,R,S,T) = x+y$. As usual, x' stands for a typical element of L, x'' for a typical element of R, y' for a typical element of S and y'' for a typical element of T.

Theorem 3.2.6 ([26, Gonshor, Theorem 3.2])**.** *The addition has the uniformity property.*

Proposition 3.2.7 ([26, Gonshor, Theorem 3.3])**.** **No** *together with the operation* $+$ *form an Abelian group with* 0 *as neutral element.*

3.2.2 Multiplication over surreal numbers

We now go on multiplication and we do the same work as for addition.

Proposition 3.2.8 ([26, Gonshor, Theorem 3.4])**.** *For all surreal numbers x and y, xy is well defined and for all surreal numbers x, y, a and b,*

$$xy = yx \qquad \textit{and} \qquad (x > y \wedge a > b) \implies ax - bx > ay - by$$

Proof. Let $\alpha, \beta, \gamma, \delta$ be the lengths of a, b, x, y respectively. We proceed by induction over $(\alpha, \beta, \gamma, \delta)$ with the lexicographic order and use the following induction hypothesis:

xy, ax, ay, bx and by are well defined and $\times$ is commutative. Moreover, if $x > y$ and $a > b$ then $ax - bx > ay - by$.

- If $\alpha = \beta = \gamma = \delta = 0$, then $x = y = a = b =$ and $ax = ay = bx = by = 0$ which is well defined and the implication holds since $x > y$ does not.

- Consider a, b, x, y of respective lengths $\alpha, \beta, \gamma, \delta$ and assume that for all u, v, w, t of respective lengths λ, μ, ν, ρ, such that $(\lambda, \nu, \mu, \rho) <_{lex} (\alpha, \beta, \gamma, \rho)$, the property holds. Note that $x', x'' \sqsubset x$ and $y', y'' \sqsubset y$. Therefore we can use the induction hypothesis on (x', y, z), (x'', y, z), (x, y', z) and (x, y'', z). Then the sets

$$L = \left\{ \begin{array}{c} x'y + xy' - x'y' \\ x''y + xy'' - x''y'' \end{array} \;\middle|\; \begin{array}{c} x', x'' \sqsubset x \\ x' < x < x'' \\ y', y' \sqsubset y \\ y' < y < y'' \end{array} \right\} \quad \text{and} \quad R = \left\{ \begin{array}{c} x'y + xy'' - x'y'' \\ x''y + xy' - x''y' \end{array} \;\middle|\; \begin{array}{c} x', x'' \sqsubset x \\ x' < x < x'' \\ y', y'' \sqsubset y \\ y' < y < y'' \end{array} \right\}$$

are well defined. We then just need to show that $L < R$. We have,

$$x'' > x' \wedge y > y' \implies x''y - x''y' > x'y - x'y' \qquad \text{(by induction hypothesis)}$$
$$x''y - x''y' > x'y - x'y' \implies x''y - x''y' + xy' > x'y - x'y' + xy' \qquad \text{(by Proposition 3.2.3)}$$

and

$$x > x' \wedge y'' > y' \implies xy' - x'y' > xy'' - x'y'' \qquad \text{(by induction hypothesis)}$$
$$xy' - x'y' > xy'' - x'y'' \implies xy' - x'y' + x'y > xy'' - x'y'' + x'y \qquad \text{(by Proposition 3.2.3)}$$

Therefore, applying the associativity of $+$ (c.f. Proposition 3.2.7), for all element element $\ell \in L$ of the form $\ell = x'y + xy' - x'y'$, we have $\ell < R$. Doing the same proof for elements of the form $x''y + xy'' - x''y''$ we get that $L < R$ so that $[L \mid R]$ is well defined. Moreover, the definition of yx gives exactly the same L and R. Therefore, $xy = yx$. The very same proof with a and x ensures that ax is well defined, as well as bx, ay and by and that all these multiplications commute.

Assume now that $x > y$ and that $a > b$.

- ➢ If $x \sqsubset y$:
  - ∵ If $a \sqsubset b$ then $ay + bx - ax$ is a left element in the definition of by. Therefore $ay + bx - ax < by$. Proposition 3.2.3 together with associativity and commutativity of $+$ (c.f. Proposition 3.2.7) ensure that we can move terms from one side to the order using a minus. Therefore we get $ax - by > ay - by$.
  - ∵ If $b \sqsubset a$ then $by + ax - bx$ is a right element of ay and again $ax - bx > ay - by$.
  - ∵ Otherwise, let c be the longest common prefix of a and b. Then $c \sqsubset a, b$ and $a > c > b$. By induction hypothesis, $ax - cx > ay - cy$ and $cx - bx > cy - by$. Then $ax > cx - cy + ay$ hence, $ax > bx - by + ay$ and finally, $ax - bx > ay - by$.
- ➢ If $y \sqsubset z$, we proceed the same way but using the cases of the definition of the multiplication we have not used yet.
- ➢ Otherwise, let z the longest common prefix of x and y. Then $z \sqsubset x, y$ and $x > z > y$. By induction hypothesis, $ax - bx > az - bz$ and $az - bz > ay - by$. By transitivity, $ax - bx > ay - by$.

□

As for addition:

Theorem 3.2.9 ([26, Gonshor, Theorem 3.5]). *The multiplication has the uniformity property.*

Proposition 3.2.10 ([26, Gonshor, Theorem 3.6]). **No** *with the operations $+$ and $\times$ is an ordered commutative ring. The neutral element for multiplication is* 1.

3.2.3 Division over surreal numbers

Although the definition of the multiplication was much more complicated than the definition of the addition, the product inverse is even more complicated. Actually, a lot of work is needed to do so. To successfully define a product inverse, we handle a new operator $\langle \cdot \rangle$ defined together with the product inverse as follows:

Definition 3.2.11. Let $x = [0, L \mid R]$ be a positive surreal number in canonical representation. In particular, $L, R > 0$[1]. We assume that the product inverse has been defined for $y \in L \cup R$ (i.e. for $0 < y \sqsubset x$). We define the product inverse of x as follows :

- Start with $\langle \rangle = 0$.
- For any $n \in \mathbb{N}$ and $y_1, \ldots, y_n \in L \cup R$, assume $\langle y_1, \ldots, y_n \rangle$ has been defined, then set

$$\langle y_1, \ldots, y_{n+1} \rangle = \frac{1}{y_{n+1}} (1 - (x - y_{n+1}) \langle y_1, \ldots, y_n \rangle)$$

- Finally define

$$\frac{1}{x} = \left[\left\{ \langle y_1, \ldots, y_n \rangle \;\middle|\; \begin{array}{c} |\{ i \mid y_i \in L \}| \in 2\mathbb{N} \\ n \in \mathbb{N} \end{array} \right\} \;\middle|\; \left\{ \langle y_1, \ldots, y_n \rangle \;\middle|\; \begin{array}{c} |\{ i \mid y_i \in L \}| \in 2\mathbb{N} + 1 \\ n \in \mathbb{N} \end{array} \right\} \right]$$

Remark 3.2.12. By induction, it can be shown that

$$\langle y_1, \ldots, y_n \rangle = \sum_{i=0}^{n-1} (-1)^i \left(\prod_{j=n-i}^{n} \frac{1}{y_j} \right) \left(\prod_{j=n-i+1}^{n} (x - y_j) \right)$$

[1]Note that if $L = \varnothing$, then $L > 0$. The same goes for R.

Proposition 3.2.13 ([26, Gonshor, Theorem 3.7]). *The product inverse is well defined over positive numbers and* $x\frac{1}{x} = 1$ *for all* x.

Proof. We show it by induction on x. Note that we initialize the induction to 1 since 1 is the shorter positive element.

- If $x = 1$, the definition gives $1/1 = [0 \mid \varnothing] = 1$ which is what is expected.
- Assume that the product inverse is well defined for all $y \sqsubset x$ such that $y > 0$. Consider the sets

$$L_k = \left\{ \langle y_1, \dots, y_n \rangle \;\middle|\; \begin{matrix} |\{ i \mid y_i \in L\}| \in 2\mathbb{N} \\ n \leq k \end{matrix} \right\} \quad \text{and} \quad R_k = \left\{ \langle y_1, \dots, y_n \rangle \;\middle|\; \begin{matrix} |\{ i \mid y_i \in L\}| \in 2\mathbb{N}+1 \\ n \leq k \end{matrix} \right\}$$

We show by induction on k that $xL_k < 1 < xR_k$.

➢ For $k = 0$, $L_0 = \{\langle\rangle\} = \{0\}$ and $R_0 = \varnothing$. In particular the property is trivial.

➢ Assume $L_k < R_k$ for some $k \in \mathbb{N}$. Let $y_1, \dots y_{k+1} \in L \cup R$. Note that

$$x \langle y_1, \dots, y_{k+1} \rangle = \frac{x}{y_{k+1}} \left(1 + (y_{k+1} - x) \langle y_1, \dots, y_k \rangle\right) = x \langle y_1, \dots, y_k \rangle + \frac{x}{y_{k+1}} \left(1 - x \langle y_1, \dots, y_k \rangle\right)$$

Denoting $\lambda = x \langle y_1, \dots, y_k \rangle$, then can write $x \langle y_1, \dots, y_{k+1} \rangle = 1\lambda + \frac{x}{y_{k+1}}(1 - \lambda)$.

∵ If $\langle y_1, \dots, y_{k+1} \rangle \in L_{k+1}$ and $y_{k+1} \in R$ then $\langle y_1, \dots, y_k \rangle \in L_k$ and $x/y_{k+1} < 1$. By induction hypothesis, $\lambda < 1$. Therefore, $x \langle y_1, \dots, y_{k+1} \rangle \in \left[\frac{x}{y_{k+1}} ; 1 \right)$, in particular $x \langle y_1, \dots, y_{k+1} \rangle < 1$.

∵ If $\langle y_1, \dots, y_{k+1} \rangle \in L_{k+1}$ and $y_{k+1} \in L$ then $\langle y_1, \dots, y_k \rangle \in R_k$ and $x/y_{k+1} > 1$. By induction hypothesis $\lambda > 1$, hence we must have $x \langle y_1, \dots, y_{k+1} \rangle < 1$.

∵ If $\langle y_1, \dots, y_{k+1} \rangle \in R_{k+1}$ and $y_{k+1} \in L$ then $\langle y_1, \dots, y_k \rangle \in L_k$ and $x/y_{k+1} > 1$. By induction hypothesis, $\lambda < 1$. Therefore, $x \langle y_1, \dots, y_{k+1} \rangle \in \left(1 ; \frac{x}{y_{k+1}} \right]$, in particular $x \langle y_1, \dots, y_{k+1} \rangle > 1$.

∵ If $\langle y_1, \dots, y_{k+1} \rangle \in R_{k+1}$ and $y_{k+1} \in R$ then $\langle y_1, \dots, y_k \rangle \in R_k$ and $x/y_{k+1} < 1$. By induction hypothesis, $\lambda > 1$, hence we must have $x \langle y_1, \dots, y_{k+1} \rangle > 1$.

By the induction principle, we then have $xL_k < 1 < xR_k$ for all $k \in \mathbb{N}$. Therefore, we also have

$$x \bigcup_{k \in \mathbb{N}} L_k < 1 < x \bigcup_{k \in \mathbb{N}} R_k$$

Using Proposition 3.2.8 (its contraposition), we then have that $\bigcup_{k \in \mathbb{N}} L_k < \bigcup_{k \in \mathbb{N}} R_k$. In particular,

$$\frac{1}{x} = \left[\left\{ \langle y_1, \dots, y_n \rangle \;\middle|\; \begin{matrix} |\{ i \mid y_i \in L\}| \in 2\mathbb{N} \\ n \in \mathbb{N} \end{matrix} \right\} \;\middle|\; \left\{ \langle y_1, \dots, y_n \rangle \;\middle|\; \begin{matrix} |\{ i \mid y_i \in L\}| \in 2\mathbb{N}+1 \\ n \in \mathbb{N} \end{matrix} \right\} \right]$$

is well defined. We now compute $x\frac{1}{x}$. By definition, and using the uniformity property of the product (Theorem 3.2.9), $x\frac{1}{x} = [S \mid T]$ where

$$S = \left\{ \begin{matrix} x'\frac{1}{x} + x \langle z_1, \dots, z_p \rangle - x' \langle z_1, \dots, z_p \rangle \\ x''\frac{1}{x} + x \langle t_1, \dots, t_p \rangle - x'' \langle t_1, \dots, t_p \rangle \end{matrix} \;\middle|\; \begin{matrix} x', x'' \sqsubset x \\ x' < x < x'' \\ \langle z_1, \dots, z_p \rangle \in \bigcup_{k \in \mathbb{N}} L_k \\ \langle t_1, \dots, t_p \rangle \in \bigcup_{k \in \mathbb{N}} R_k \\ p \in \mathbb{N} \end{matrix} \right\}$$

and

$$T = \left\{ \begin{matrix} x'\frac{1}{x} + x \langle t_1, \dots, t_p \rangle - x' \langle t_1, \dots, t_p \rangle \\ x''\frac{1}{x} + x \langle z_1, \dots, z_p \rangle - x'' \langle z_1, \dots, z_p \rangle \end{matrix} \;\middle|\; \begin{matrix} x', x'' \sqsubset x \\ x' < x < x'' \\ \langle z_1, \dots, z_p \rangle \in \bigcup_{k \in \mathbb{N}} L_k \\ \langle t_1, \dots, t_p \rangle \in \bigcup_{k \in \mathbb{N}} R_k \\ p \in \mathbb{N} \end{matrix} \right\}$$

Note that $(x-x')\langle z_1,\dots,z_p\rangle = 1-x'\langle z_1,\dots,z_p,x'\rangle$ and that $\langle z_1,\dots,z_p,x'\rangle \in R_{p+1}$ for any $\langle z_1,\dots,z_p\rangle \in L_p$ and $x' \sqsubset x$ such that $0 < x' < x$. Hence, for $x' \neq 0$,

$$x'\frac{1}{x} + x\langle z_1,\dots,z_p\rangle - x'\langle z_1,\dots,z_p\rangle = 1 + x'(\frac{1}{x} + 1 - \langle z_1,\dots,z_p,x'\rangle) < 1$$

and if $x' = 0$,
$$x'\frac{1}{x} + x\langle z_1,\dots,z_p\rangle - x'\langle z_1,\dots,z_p\rangle = x\langle z_1,\dots,z_p\rangle < 1$$

Similarly, we get all the other inequalities to ensure $S < 1 < T$. The only prefix of 1 being 0 we now now that $[S \mid T] \in \{0,1\}$. But for $x' = 0$ and $p = 0$ we get that $0(1/x) + x0 - 00$ is in S. Finally, $[S \mid T] \neq 0$ and

$$x\frac{1}{x} = 1$$

□

Corollary 3.2.14. **No** *with the operations* $+$ *and* $\times$ *is an ordered field.*

Remark 3.2.15. From a set theoretic point of view, **No** is not a field since it is a proper class. It would be more accurate to say that **No** is a class field but for the sake of simplicity we omit this detail.

The previous corollary is the actual statement of [26, Gonshor, Theorem 3.7] whose proof is essentially the one of the previous proposition.
In the previous proof, the fact that we were using the canonical representation of x is just hear to be able to define properly the product inverse. Now that it is done, the very same work on any representation $x = [L \mid R]$ with $L, R > 0$ gives again a definition for the product inverse. Therefore,

Corollary 3.2.16. *The product inverse has the uniformity property in the following sense: For all positive* $x = [L \mid R]$ *such that* $L, R > 0$ *we have*

$$\frac{1}{x} = \left[\left\{ \langle y_1,\dots,y_n\rangle \;\middle|\; \begin{array}{c} |\{i \mid y_i \in L\}| \in 2\mathbb{N} \\ n \in \mathbb{N} \end{array} \right\} \;\middle|\; \left\{ \langle y_1,\dots,y_n\rangle \;\middle|\; \begin{array}{c} |\{i \mid y_i \in L\}| \in 2\mathbb{N}+1 \\ n \in \mathbb{N} \end{array} \right\}\right]$$

3.3 Normal form

3.3.1 Interlude : Hahn series

Recall that a group G is **divisible** is for any $n \in \mathbb{N}^*$ and any $g \in G$, there is some $h \in G$ such that $nh = g$. Given that, let $\mathbb{K}$ be a field, and let G be a divisible ordered Abelian group.

Definition 3.3.1 (Hahn series [27])**.** The Hahn series (obtained from $\mathbb{K}$ and G) are formal power series of the form $s = \sum_{g\in S} a_g t^g$, where S is a well-ordered subset of G and $a_g \in \mathbb{K}^*$. We may also write $s = \sum\limits_{g\in G} a_g t^g$ and say that S is the support of s denoted $\operatorname{supp}(s) = \{g \in S \mid a_g \neq 0\}$ and the length of s is the order type of $S = \operatorname{supp}(s)$.
We write $\mathbb{K}((G))$ for the set of Hahn series with coefficients in K and terms corresponding to elements of G.

Definition 3.3.2 (Operations on $\mathbb{K}((G))$)**.** The operations on $\mathbb{K}((G))$ are defined in the expected way, considering them as formal power series: Let

$$s = \sum_{g\in S} a_g t^g \qquad \text{and} \qquad s' = \sum_{g\in S'} a'_g t^g$$

where S, S' are well ordered.

- $s + s' = \sum_{g\in S\cup S'} \left(a_g + a'_g\right) t^g$, where $a_g = 0$ if $g \notin S$, and $a'_g = 0$ if $g \notin S'$.
- $s \cdot s' = \sum_{g\in T} b_g t^g$, where $T = \{g_1 + g_2 \mid g_1 \in S \wedge g_2 \in S'\}$, and for each $g \in T$, we set

$$b_g = \sum_{g_1\in S, g_2\in S' | g_1+g_2=g} b_{g_1} \cdot b_{g_2}$$

Remark 3.3.3. Note that the operations are well-defined because of Propositions 2.4.2 for addition and 2.4.3 for multiplication. Namely, these propositions ensure that the set of exponents is still well-ordered.

It is common work to check that the operations defined above give $\mathbb{K}((G))$ a ring structure. It the same reasons why polynomials with coefficients in $\mathbb{K}$ have a ring structure. However, we can go further: We actually have a field. We then call the structures of the form $\mathbb{K}((G))$ Hahn fields.

Proposition 3.3.4. *$\mathbb{K}((G))$ with operations $+$ and $\times$ is field and the product inverse is given by the following: For $s = \sum_{s\in S} a_g t^g$ with non-empty support S,*

$$\frac{1}{s} = \frac{1}{a_{g_0}} t^{-g_0} \left(1 + \sum_{g\in\langle S-g_0\rangle} \left(\sum_{k\in\mathbb{N}^*} (-1)^k \sum_{\substack{g_1,\ldots,g_k\,\in\, S\setminus\{g_0\} \\ g_1+\cdots+g_k-kg_0=g}} \frac{a_{g_1}\cdots a_{g_k}}{{a_{g_0}}^k} \right) t^g \right)$$

where g_0 is the minimum element of S and $\langle S - g_0\rangle$ is the monoid generated by $S - g_0$ in G.

Proof. Let $s = \sum_{s\in S} a_g t^g$ with non-empty support S and g_0 being its minimum element. We have that $S - g_0 \geq 0$. Applying Proposition 2.4.5, $\langle S - g_0\rangle$ is well ordered. Therefore, for any $g \in \langle S - g_0\rangle$, there are finitely many $k \in \mathbb{N}$ and finitely many $g_1, \ldots, g_k \in S \setminus \{g_0\}$ such that $g_1 + \cdots + g_k - kg_0 = g$. That is, for any $g \in \langle S - g_0\rangle$, the sum

$$\sum_{k\in\mathbb{N}^*} (-1)^k \sum_{\substack{g_1,\ldots,g_k\,\in\, S\setminus\{g_0\} \\ g_1+\cdots+g_k-kg_0=g}} \frac{a_{g_1}\cdots a_{g_k}}{{a_{g_0}}^k}$$

contains finitely many term and therefore is well-defined. Thus the element s' defined by

$$s' = \frac{1}{a_{g_0}} t^{-g_0} \left(1 + \sum_{g\in\langle S-g_0\rangle} \left(\sum_{k\in\mathbb{N}^*} (-1)^k \sum_{\substack{g_1,\ldots,g_k\,\in\, S\setminus\{g_0\} \\ g_1+\cdots+g_k-kg_0=g}} \frac{a_{g_1}\cdots a_{g_k}}{{a_{g_0}}^k} \right) t^g \right)$$

is well defined and $s' \in \mathbb{K}((G))$. It remain to check that $s's = 1$.

$$\begin{aligned}
s's &= \left(1 + \sum_{g\in\langle S-g_0\rangle} \left(\sum_{k\in\mathbb{N}^*} (-1)^k \sum_{\substack{g_1,\ldots,g_k\,\in\, S\setminus\{g_0\} \\ g_1+\cdots+g_k-kg_0=g}} \frac{a_{g_1}\cdots a_{g_k}}{{a_{g_0}}^k} \right) t^g \right) \left(1 + \sum_{g\in S\setminus\{g_0\}} \frac{a_g}{a_{g_0}} t^{g-g_0} \right) \\
&= 1 + \sum_{g\in\langle S-g_0\rangle} \left(\sum_{k\in\mathbb{N}^*} (-1)^k \sum_{\substack{g_1,\ldots,g_k\,\in\, S\setminus\{g_0\} \\ g_1+\cdots+g_k-kg_0=g}} \frac{a_{g_1}\cdots a_{g_k}}{{a_{g_0}}^k} \right) t^g + \sum_{g\in S\setminus\{g_0\}} \frac{a_g}{a_{g_0}} t^{g-g_0} \\
&\quad + \left(\sum_{g\in\langle S-g_0\rangle} \left(\sum_{k\in\mathbb{N}^*} (-1)^k \sum_{\substack{g_1,\ldots,g_k\,\in\, S\setminus\{g_0\} \\ g_1+\cdots+g_k-kg_0=g}} \frac{a_{g_1}\cdots a_{g_k}}{{a_{g_0}}^k} \right) t^g \right) \left(\sum_{g\in S\setminus\{g_0\}} \frac{a_g}{a_{g_0}} t^{g-g_0} \right) \\
&= 1 + \sum_{g\in\langle S-g_0\rangle} \left(\sum_{k\in\mathbb{N}\setminus\{0,1\}} (-1)^k \sum_{\substack{g_1,\ldots,g_k\,\in\, S\setminus\{g_0\} \\ g_1+\cdots+g_k-kg_0=g}} \frac{a_{g_1}\cdots a_{g_k}}{{a_{g_0}}^k} \right) t^g \\
&\quad + \sum_{g\in\langle S-g_0\rangle+(S-g_0)} \left(\sum_{k\in\mathbb{N}^*} (-1)^k \sum_{\substack{g_1,\ldots,g_{k+1}\,\in\, S\setminus\{g_0\} \\ g_1+\cdots+g_{k+1}-(k+1)g_0=g}} \frac{a_{g_1}\cdots a_{g_{k+1}}}{{a_{g_0}}^{k+1}} \right) t^g \\
&= 1 + \sum_{g\in\langle S-g_0\rangle} \left(\sum_{k\in\mathbb{N}\setminus\{0,1\}} (-1)^k \sum_{\substack{g_1,\ldots,g_k\,\in\, S\setminus\{g_0\} \\ g_1+\cdots+g_k-kg_0=g}} \frac{a_{g_1}\cdots a_{g_k}}{{a_{g_0}}^k} \right) t^g \\
&\quad - \sum_{g\in\langle S-g_0\rangle\setminus\{0\}} \left(\sum_{k\in\mathbb{N}\setminus\{0,1\}} (-1)^k \sum_{\substack{g_1,\ldots,g_k\,\in\, S\setminus\{g_0\} \\ g_1+\cdots+g_k-kg_0=g}} \frac{a_{g_1}\cdots a_{g_k}}{{a_{g_0}}^k} \right) t^g \\
&= 1 + \sum_{k\in\mathbb{N}\setminus\{0,1\}} (-1)^k \sum_{\substack{g_1,\ldots,g_k\,\in\, S\setminus\{g_0\} \\ g_1+\cdots+g_k-kg_0=0}} \frac{a_{g_1}\cdots a_{g_k}}{{a_{g_0}}^k} \\
&= 1 \qquad \text{(since } S - g_0 > 0\text{)}
\end{aligned}$$

□

Definition 3.3.5. Provided that $\mathbb{K}$ is ordered by $<$, the Hahn field $\mathbb{K}((G))$ can be given an ordering that extends $<$ as follows: Let $x = \sum_{g\in S} a_g t^g$ and $y = \sum_{g\in S'} b_g t^g$. Let T be longest common initial segment of S and S' and

$$g_x = \begin{cases} \min\{g \in S \mid g \notin T\} & \text{if } S \neq T \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g_y = \begin{cases} \min\{g \in S' \mid g \notin T\} & \text{if } S' \neq T \\ 0 & \text{otherwise} \end{cases}$$

$$r_x = \begin{cases} a_{g_x} & \text{if } S \neq T \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad r_y = \begin{cases} a_{g_y} & \text{if } S' \neq T \\ 0 & \text{otherwise} \end{cases}$$

We set

$$x < y \iff \begin{cases} & r_x \leq 0 \leq r_y \wedge r_x < r_y \\ \text{or} & (g_x, -r_x) <_{lex} (g_y, -r_y) \ \wedge \ r_x, r_y < 0 \\ \text{or} & (g_x, r_x) >_{lex} (g_y, r_y) \ \wedge \ r_x, r_y > 0 \end{cases}$$

Remark 3.3.6. This ordering give $\mathbb{K}((G))$ a structure of ordered field.

Remark 3.3.7. The definition of the order may seem quite complicated by if we write $x = \sum_{g\in S\cup S'} a_g t^g$ and $y = \sum_{g\in S\cup S'} b_g t^g$ allowing $a_g = 0$ for $g \in S' \setminus S$ and $b_g = 0$ for $g \in S \setminus S'$, the definition of the order actually become the usual lexicographic order.

Hahn fields inherits a lot of from the structure of the coefficient field. In particular if $\mathbb{K}$ is algebraically closed, and if G is some divisible (i.e. for any $n \in \mathbb{N}$ and $g \in G$ there is some $g' \in G$ such that $ng' = g$) ordered Abelian group, then the corresponding Hahn field is also algebraically closed.

Theorem 3.3.8 (Generalized Newton-Puiseux Theorem, Maclane [36])**.** *Let G be a divisible ordered Abelian group, and let $\mathbb{K}$ be a field that is algebraically closed of characteristic 0. Then $\mathbb{K}((G))$ is also algebraically closed.*

Hahn fields are also ordered fields. When dealing with such field, one may often wonder if this fields are real closed. As noticed in [1], we can deduce from the previous theorem the following:

Corollary 3.3.9. *Let G be a divisible ordered Abelian group, and let $\mathbb{K}$ be a field that is real closed (i.e -1 is not a square and $\mathbb{K}(i)$ is algebraically closed) of characteristic 0. Then $\mathbb{K}((G))$ is also real closed.*

Proof. $\mathbb{K}$ is real closed. That is to say that -1 is not a square in $\mathbb{K}$ and that $\mathbb{K}[i]$ is algebraically closed. Notice that $\mathbb{K}[i]((G)) = (\mathbb{K}((G))))[i]$. Therefore, Theorem 3.3.8 ensures that $(\mathbb{K}((G)))[i]$ is algebraically closed. Also, -1 is not a square in $\mathbb{K}((G))$. Therefore, $\mathbb{K}((G))$ is real closed. ☐

As well as we defined $\mathbf{No}_\alpha$ restricting the length od the surreal numbers, we can restrict the length of the series. We then introduce

Definition 3.3.10 (Alling, [1])**.** $\mathbb{K}((G))_\lambda$ is set of elements from $\mathbb{K}((G))$ whose support as order type less than the ordinal number λ.

$$\mathbb{K}((G))_\lambda = \{x \in \mathbb{K}((G)) \mid \operatorname{supp} x < \lambda\}$$

Notation. For the sake of readability, we may forget the parenthesis and use the notation $\mathbb{K}((G))_\lambda = \mathbb{K}_\lambda^G$. Finally, if $(G_i)_{i\in I}$ is a family of Abelian groups, we also denote

$$\mathbb{K}_\lambda^{(G_i)_{i\in I}} = \bigcup_{i\in I} \mathbb{K}_\lambda^{G_i}$$

3.3.2 The ω-map : Archimedean classes

We have introduced Hahn series in the previous section. Now we show that we can associate any surreal number with such a series. More precisely, we will see each surreal number has a normal form that can be expressed as a Hahn series. This series will match the Cantor normal form for ordinal numbers seen as surreal numbers. Namely, $\frac{1}{\omega}$ will play the role of the variable t in Hahn series. To do so, we need to define what is ω^a for any surreal number a.

Definition 3.3.11. We define the following relations for a and b two surreal numbers:

- $a \asymp b$ *iff* there is some natural number n such that $n|a| \geq |b|$ and $n|b| \geq |a|$. We say that a and b have the same order of magnitude.
- $a \prec b$ *iff* for all natural number n, $n|a| < |b|$. We say that b a higher order of magnitude than a.
- $a \preceq b$ *iff* $a \prec b$ or $a \asymp b$. We say that b as at least the same order of magnitude as a.

- $a \sim b$ *iff* $a - b \prec 1$. We say that a and b are equivalent.

The following is immediate:

Lemma 3.3.12. *The relation $\preceq$ is a preorder and $\asymp$ and $\prec$ are the associated equivalence relation and strict preorder respectively.*

Now that we have defined an equivalence relation, we can wonder what are the equivalences classes and if there is a "canonical" representative of each class. First, the classes are obviously, by definition, the Archimedean classes. Indeed, two surreals are in the same class if and only if there is a multiple of the first one that is greater than the second one and respectively. Now, for the "canonical" part, in our context, if there is a shorter surreal number in each class it would be nice. In turns out that it is the case.

Theorem 3.3.13 ([26, Gonshor, Theorem]). *For any non-zero surreal number a, there is a unique shortest positive element x such that $x \asymp a$. More precisely, if $y \asymp a$ and $y > 0$ then $x \sqsubseteq y$.*

Remark 3.3.14. Note that we cannot require x to be the unique element with minimal length since $-x$ is in the same class and has the same length. However, if we ensure $x > 0$, we fully characterize the minimum element with respect to the partial order $\sqsubseteq$.

Proof. Let $x \asymp a$ be positive with minimal length. Let $y > 0$ such that $a \asymp y$, or equivalently $x \asymp y$. Let z be the longest common prefix of x and y. Then $z \asymp x$ and $z > 0$ since both x and y are positive. By minimality of x, we have $z = x$ and therefore $x \sqsubseteq y$. □

We now try to define explicitly the shorter positive elements of each class. We will index them with all the surreal numbers.

Definition 3.3.15. We define for all $x = [x' \mid x'']$ in canonical representation,

$$\omega^x = \left[0, \mathbb{R}_+^* \omega^{x'} \;\middle|\; \mathbb{R}_+^* \omega^{x''}\right]$$

An element of the form ω^x will be called a **monomial**.

Example 3.3.16. As an example, we immediately have $\omega^0 = 1, \omega^1 = \omega$.

Lemma 3.3.17 ([26, Gonshor, Theorem 5.2]). *ω^x is well-defined for all surreal number x and for any x, y surreal numbers,*

$$x < y \implies \omega^x \prec \omega^y$$

Moreover, $x \mapsto \omega^x$ has the uniformity property

Theorem 3.3.18 ([26, Gonshor, Theorem 5.3] and [18, Theorem 19]). *A surreal number x is of the form $x = \omega^a$ if and only if it is the shortest positive element in its Archimedean class.*

Proof. (NC $\Rightarrow$) Assume that $x = \omega^a$ and that $y > 0$ is such that $y \asymp x$. By Lemma 3.3.17, we have $y \prec \omega^{a''}$ and $y \succ \omega^{a'}$ when writing $a = [a' \mid a'']$ in canonical representation. Therefore $0, \mathbb{R}_+^* \omega^{a'} < y < \omega^{a''}$. By definition of $[\cdot \mid \cdot]$, $x \sqsubseteq y$.

(SC $\Leftarrow$) We proceed by induction on x. First, for $x = 1$, $x = \omega^0$ and the property is true. Now, let x be a positive surreal number which is the shortest in its Archimedean class. Assume that for any $y \sqsubset x$, if y is the shortest in its Archimedean class then y is of the form ω^{a_y}. By minimality of x, for all $y \sqsubset x$, $y \prec x$ or $y \succ x$. Applying Theorem 3.3.13, for all $y \sqsubset x$, we get that for all $y \sqsubset x$ there is a unique $z \sqsubset y$ of minimal length such that $z_y \asymp y$. By minimality of x, for all $y \sqsubset x$, $z_y = \omega^{a_y}$ for some surreal number a_y. Let

$$L = \{a_y \mid y \sqsubset x \quad y < x\} \qquad \text{and} \qquad R = \{a_y \mid y \sqsubset x \quad y > x\}$$

We claim that $L < R$. Indeed, if there is $\ell \in L$ and $r \in R$ such that $\ell \geq r$, then $\omega^{a_\ell} \prec x \prec \omega^{a_r} \prec \omega^{a_\ell}$. We then consider $a = [L \mid R]$. Note that by definition, Lemma 3.3.17 and more precisely the uniformity property,

$$\omega^a = \left[0, \mathbb{R}_+^* \omega^L \mid \mathbb{R}_+^* \omega^R\right]$$

Since $0, \mathbb{R}_+^* \omega^L < x < \mathbb{R}_+^* \omega^R$, we then have $\omega^a \sqsubseteq x$. If $\omega^a \neq x$, then $a \in L \cup R$ which is impossible. Therefore $x = \omega^a$.

□

The previous theorem ensures that the ω-map is a parametrization of all the canonical representative of each class. This is our first example of surreal substructure that will be studied in Section 3.5. This elements will be our fundamental bricks to build the normal form of surreal numbers. Since we already announced that such a normal form must be a generalization of the Cantor's normal form, we may expect that ω^α as an ordinal exponentiation is the same as ω^α as the ω-map applied to the ordinal number α seen as a surreal number.

Theorem 3.3.19 ([26, Gonshor, Theorem 5.4] and [18, Conway, Theorem 20])**.** *For all surreal numbers a and b, $\omega^a\omega^b = \omega^{a+b}$ and for all ordinal α, ω^α as an ordinal exponentiation is the same as ω^α as the ω-map applied to the ordinal number α seen as a surreal number.*

3.3.3 Normal form for surreal numbers

Having defined the ω-map, we now define formal power series thanks to the monomials ω^x.

Definition 3.3.20 ([26, Gonshor, page 59])**.** Let $(a_i)_{i<\nu}$ be an ordinal-length decreasing sequence of surreal numbers and $(r_i)_{i<\nu}$ be non-zero real numbers. We define by transfinite induction :

- $\sum\limits_{i<0} r_i\omega^{a_i} = 0$
- If $\nu = \nu' + 1$ then $\sum\limits_{i<\nu} r_i\omega^{a_i} = \sum\limits_{i<\nu'} r_i\omega^{a_i} + r_{\nu'}\omega^{a_{\nu'}}$
- If ν is a limit ordinal then

$$\sum_{i<\nu} r_i\omega^{a_i} = \left[\left\{\sum_{i<\nu'} r_i\omega^{a_i} + (r_{\nu'} - \varepsilon)\omega^{a_{\nu'}} \;\middle|\; \begin{matrix}\nu' < \nu \\ \varepsilon \in \mathbb{R}_+^*\end{matrix}\right\} \;\middle|\; \left\{\sum_{i<\nu'} r_i\omega^{a_i} + (r_{\nu'} + \varepsilon)\omega^{a_{\nu'}} \;\middle|\; \begin{matrix}\nu' < \nu \\ \varepsilon \in \mathbb{R}_+^*\end{matrix}\right\}\right]$$

If $x = \sum\limits_{i<\nu} r_i\omega^{a_i}$, we will call this writing the **normal form** of x. An element of the form $r\omega^a$ with $a \in \mathbf{No}$ and $r \in \mathbb{R}$ will be called a **term**. In particular, a monomial is a term. Finally if $x = \sum\limits_{i<\nu} r_i\omega^{a_i}$ we denote $\nu(x) = \nu$ the length of the series in the normal form of x.

Proposition 3.3.21 ([26, Gonshor, Theorem 5.5])**.** *The writing $\sum\limits_{i<\nu} r_i\omega^{a_i}$ is well defined for all ordinal-length decreasing sequence of surreal number $(a_i)_{i<\nu}$ and for all non-zero real numbers $(r_i)_{i<\nu}$. Moreover, for all $\nu' < \nu$,*

$$\sum_{i<\nu} r_i\omega^{a_i} - \sum_{i<\nu'} r_i\omega^{a_i} \prec \omega^{a_j} \iff j < \nu'$$

Finally, looking these elements as members of the Hahn field $\mathbb{R}((\mathbf{No}))$, the function $\sum\limits_{i<\nu} r_i\omega^{a_i} \mapsto \sum\limits_{i<\nu} r_it^{-a_i}$ is an ordered set isomorphism. In particular, the order[2] *for Hahn series as in Definition 3.3.5 corresponds to the order of surreal numbers.*

We now show that this writing is a normal form for all surreal numbers. To that purpose, we state the following technical lemma:

Lemma 3.3.22 ([26, Gonshor, page 63])**.** *For all ordinal-length decreasing sequence of surreal number $(a_i)_{i<\nu}$ and for all non-zero real numbers $(r_i)_{i<\nu}$, $\left|\sum\limits_{i<\nu} r_i\omega^{a_i}\right|_{+-} \geq \nu$*

Proof. By definition, we can see that $\sum\limits_{i<\nu'} r_i\omega^{a_i} \sqsubset \sum\limits_{i<\nu} r_i\omega^{a_i}$ whenever $\nu' < \nu$. The result simply follows by induction. □

As expressed by Gonshor, this lower bound is not accurate at all and we can do much much better. However, its is sufficient for now.

Theorem 3.3.23 ([26, Gonshor, Theorem 5.6] and [18, Conway, Theorem 21])**.** *Every surreal number can be uniquely expressed in the way $x = \sum\limits_{i<\nu} r_i\omega^{a_i}$.*

Proof. Assume that x cannot be written is that way. We define $x_\alpha = \sum\limits_{i<\alpha} r_i\omega^{a_i}$ for all ordinal α as follows:

[2] Actually, Gonshor stated that the order was the lexicographic order over the pairs (a_i, r_i), which is true if and only if the r_is are positive. For instance $-\omega < 1$ but the respective pairs are $(1, -1)$ and $(0, 1)$ so that $(1, -1) > (0, 1)$. What suggests Gonshor remains true if the r_is are allowed to be 0 in the sens of Remark 3.3.7.

- $x_0 = 0$
- If $\alpha = \beta + 1$, a_β be the unique surreal number such that $\omega^{a_\beta} \asymp x - x_\beta$. It is given by Theorem 3.3.18. Let $r_\beta = \sup\{r \in \mathbb{R} \mid x - x_\beta - r\omega^{a_\beta} \geq 0\}$. Then $x - x_\beta \sim r_\beta \omega^{a_\beta}$. Set $x_\alpha = x_\beta + r_\beta \omega^{a_\beta}$.
- If α is a limit ordinal, just apply the Definition 3.3.20 to get x_α.

By Lemma 3.3.22, $|x_\alpha|_{+-} \geq \alpha$ for all ordinal α. On the other hand, by definition, for all limit ordinal α,

$$x_\alpha = \left[\left\{ \sum_{i<\nu'} r_i\omega^{a_i} + (r_{\nu'} - \varepsilon)\omega^{a_{\nu'}} \;\middle|\; \begin{array}{c} \nu' < \nu \\ \varepsilon \in \mathbb{R}_+^* \end{array} \right\} \middle| \left\{ \sum_{i<\nu'} r_i\omega^{a_i} + (r_{\nu'} + \varepsilon)\omega^{a_{\nu'}} \;\middle|\; \begin{array}{c} \nu' < \nu \\ \varepsilon \in \mathbb{R}_+^* \end{array} \right\}\right]$$

and by definition of sup,

$$\left\{ \sum_{i<\nu'} r_i\omega^{a_i} + (r_{\nu'} - \varepsilon)\omega^{a_{\nu'}} \;\middle|\; \begin{array}{c} \nu' < \nu \\ \varepsilon \in \mathbb{R}_+^* \end{array} \right\} < x < \left\{ \sum_{i<\nu'} r_i\omega^{a_i} + (r_{\nu'} + \varepsilon)\omega^{a_{\nu'}} \;\middle|\; \begin{array}{c} \nu' < \nu \\ \varepsilon \in \mathbb{R}_+^* \end{array} \right\}$$

Therefore, by definition of $[\cdot \mid \cdot]$, we have $x_\alpha \sqsubseteq x$. Hence,

$$\forall \alpha \in \mathbf{Lim} \qquad \alpha \leq |x_\alpha|_{+-} \leq |x|_{+-}$$

which is obviously a contradiction.
Now, for uniqueness, using Proposition 3.3.21, the ordering of surreal number coincides with the order over the Hahn series. Therefore, two distinct series cannot represent the same surreal number.

□

The normal form is also very convenient to handle operations. Indeed we have:

Theorem 3.3.24 ([26, Gonshor, Lemma 5.5, Theorems 5.7 and 5.8]). *The function* $\sum_{i<\nu} r_i\omega^{a_i} \mapsto \sum_{i<\nu} r_i t^{-a_i}$ *is an ordered field isomorphism.*

The previous theorem gives us a new way to see surreal number. Indeed, as the operations can be performed in a formal way, we can use usual algebra with surreal numbers. In the following chapters, we will be particularly interested in using this form of the surreal numbers.

Remark 3.3.25. Let λ be an ε-number. We see it as a surreal number. The normal form of λ is ω^λ, is other words, λ is already in normal form.

Looking at the normal form enables us to consider some special cases:

Definition 3.3.26. A surreal number a in normal form $a = \sum_{i<\nu} r_i\omega^{a_i}$ is

- **purely infinite** if for all $i < \nu$, $a_i > 0$. If $\mathbb{K} \subseteq \mathbf{No}$ is a subfield of $\mathbf{No}$, we denote $\mathbb{K}_\infty$ the set (or class) of purely infinite numbers in $\mathbb{K}$. We also denote $\mathbb{K}_\infty^+$ the set (or class) of non-negative purely infinite numbers.
- **infinitesimal** if for all $i < \nu$, $a_i < 0$ (or equivalently if $a \prec 1$).
- **appreciable** if for all $i < \nu$, $a_i \leq 0$ (or equivalently if $a \preceq 1$).

If $\nu' \leq \nu$ is the first ordinal such that $a_i \leq 0$, then $\sum_{i<\nu'} r_i\omega^{a_i}$ is called the **purely infinite part** of a. Similarly, if $\nu' \leq \nu$ is the first ordinal such that $a_i < 0$, $\sum_{\nu' \leq i < \nu} r_i\omega^{a_i}$ is called the **infinitesimal part** of a.

From Gonshor's arguments, when looking at the signs sequences, a purely infinite number is is a surreal number x of limit ordinal length such that for all ordinal α, if the sign $x(\alpha)$ exists, $x(\alpha+1) = x(\alpha)$, that is, no minus follows directly a plus and respectively. An infinitesimal number is such that it start with either $(+)(-)^\omega$ or $(-)(+)^\omega$. Finally, an appreciable number is such that it starts with a finite number of pluses or a finite number of minuses.

3.3.4 Signs sequence and normal form

We have introduced the normal form, we now explain how to effectively get the normal form from the signs sequence and conversely. In [26], Gonshor shows how to get the signs sequence from the the series expression. More precisely, if we are given the signs sequences for the exponents and coefficients, he provides a procedure to get back the signs sequence.

Definition 3.3.27 (Reduced signs sequence, Gonshor, [26])**.** Let $x = \sum_{i<\nu} r_i\omega^{a_i}$ be a surreal number. The reduced signs sequence of a_i, denoted a_i° is inductively defined as follows :

- $a_0^\circ = a_0$
- For $i > 0$, if $a_i(\delta) = -$ and if there is there is $j < i$ such that for $\gamma \leq \delta$, $a_j(\gamma) = a_i(\gamma)$, then we discard the minus in position δ in the signs sequence of a_i.
- If $i > 0$ is a non-limit ordinal and $(a_{i-1})_-$ (as a signs sequence) is a prefix of a_i, then we discard this minus after a_{i-1} if r_{i-1} is not a dyadic rational number.

a_i° is the signs sequence obtained when copying a_i omitting the discarded minuses. To give an intuition about what is going on, we forget the minuses that have already been treated before. We just keep the new one brought by a_i expected when this minus is implicit: If $a_i \sqsubset a_{i+1}$, then, since the sequence must be decreasing, we already know that $a_i(-) \sqsubseteq a_i$; The first new minus does not bring any information and can be discarded for "simplicity".

Theorem 3.3.28 ([26, Gonshor, Theorems 5.11 and 5.12])**.** *For a surreal number a,*

- *The signs sequence of ω^a is as follows : we start with a plus and the for any ordinal $\alpha < |a|$ we add $\omega^{|a[:\alpha]|_+ +1}$ occurrences of $a(\alpha)$.*
- *The signs sequence of $\omega^a n$ is the signs sequence of ω^a followed by $\omega^{|a|_+}(n-1)$ pluses.*
- *The signs sequence of $\omega^a \dfrac{1}{2^n}$ is the signs sequence of ω^a followed by $\omega^{|a|_+}n$ minuses.*
- *The signs sequence of $\omega^a r$ for r a positive real is the signs sequence of ω^a to which we add each sign of r $\omega^{|a|_+}$ times excepted the first plus which is omitted.*
- *The signs sequence of $\omega^a r$ for r a negative real is the signs sequence of $\omega^a(-r)$ in which we change every plus in a minus and conversely.*
- *The signs sequence of $\sum_{i<\nu} r_i\omega^{a_i}$ is the juxtaposition of the signs sequences of the $\omega^{a_i^\circ} r_i$*

We can give the transformation in the other direction. It will be very convenient to be sure that theses point of view are computationally equivalent. The procedure is given by Algorithm 2. By construction, the series that it returns admits the input signs sequence as a signs expansion. By uniqueness of both of the writings, the algorithm is correct. It uses the procedure of un-reduction given by the Algorithm 1.

Algorithm 1 Un-reduction : unReduce(a,A,R)

Inputs : a, a signs sequence, A, R lists of exponents and coefficients
Require: A be a decreasing sequence
Require: A and R have the same length $|A| = |R|$
Ensure: The returned value, y is such that $a = y^\circ$ in the context of the reduction of exponents in $\sum_{i<|A|} R[i]\omega^{A[i]} + \omega^y$

$i \leftarrow |A|$
$l = \min\{\alpha \mid a(\alpha) = -\}$ ($= |a|_{+-}$ if empty) ▷ We have $a = (+)^l a_{i,1}$.
$a_{i,1} \leftarrow a[l:]$
if i is a limit ordinal **then**
  $E \leftarrow \{\alpha \mid \exists j < i \quad \forall \beta \leq \alpha \quad \forall j < k < i \quad A[k](\beta) = A[j](\beta)\}$
  for all $\alpha \in E$ **do**
    $j_\alpha \leftarrow \min\{j \mid \forall \beta \leq \alpha \quad \forall j < k < i \quad A[k](\beta) = A[j](\beta)\}$
    $aux(\alpha) \leftarrow A[j_\alpha](\alpha)$
  end for
  if aux has a prefix such that its "+"s have order type l **then**
    $u \leftarrow$longest prefix of aux such that its "+"s have order type l
    return $ua_{i,1}$ ▷ Concatenation of strings
  else
    return $aux(+)^{l-\text{orderType}(\text{"+"s of } aux)} a_{i,1}$ ▷ Concatenation of strings
  end if
else
  $a'_{i-1} \leftarrow$shortest prefix of $A[-1]$ such that its "+"s have order type l ▷ $A[-1]$ is the last element of A
  $p = |a'_{i-1}|_{+-}$
  $q = \min\{\alpha \geq p \mid A[-1](\alpha) \neq -\}$
  $m_i = q - p$
  if $A[-1](q)$ does not exists and $R[-1]$ is not dyadic **then**
    $n_i \leftarrow m_i + 1$
  else
    $n_i \leftarrow m_i$
  end if
  return $a'_{i-1}(-)^{n_i} a_{i,1}$
end if

To show the correctness of Algorithm 1 we state the following:

Lemma 3.3.29. *Let $(a_i)_i$ be a decreasing sequence of surreal numbers and $(r_i)_i$ be real numbers. Then for every i, a_i is of the form of a concatenation $a_{i,0}a_{i,1}$ such that every "$-$" of $a_{i,0}$ is discarded and none of the ones of $a_{i,1}$ and such that $a_{i,0}$ is maximal for this property (that is $a_{i,1}$ does not starts with a "$+$").*

Proof. If the minus in position δ is discarded by some a_j with $j < i$, then if there is a minus in position $\delta' < \delta$, it must be discarded by the same a_j. If the position δ was the last position of a_i, then $a_i = a_{i-1}-$ and so a_{i-1} discards the minus in position δ'. □

We then have immediately the following corollary.

Corollary 3.3.30. *Let $(a_i)_i$ be a decreasing sequence of surreal numbers and $(r_i)_i$ be real numbers. For every i we have $a_i^\circ = (+)^{l_i} a_{i,1}$ with $(+)^{l_i}$ being the maximal prefix of a_i° containing only pluses.*

Corollary 3.3.31. *Let $(a_i)_i$ be a decreasing sequence of surreal numbers and $(r_i)_i$ be real numbers. Let $a'_{i,0}$ be the the smallest prefix of $a_{i,0}$ such that the order type of its "$+$"s is l_i. There is n_i an ordinal such that $a_{i,0} = a'_{i,0}(-)^{n_i}$.*

Lemma 3.3.32. *Let $(a_i)_i$ be a decreasing sequence of surreal numbers and $(r_i)_i$ be real numbers. Let i be non-limit. Write $a_{i-1} = a'_{i-1}(-)^{m_i}a''_{i-1}$ where the "$+$"s of a'_{i-1} have order type l_i and a''_{i-1} does not start with a "$-$". Then $a'_{i,0} = a'_{i-1}$ $\quad n_i = \begin{cases} m_i + 1 & \text{if } a_{i-1} = a'_{i-1}(-)^{m_i} \text{ and } r_{i-1} \text{ is not dyadic} \\ m_i & \text{otherwise} \end{cases}$*

Proof. Since $a_i < a_{i-1}$ we need a'_{i-1} to be a prefix of a_i. Otherwise, if b is the longest common prefix, and is shorter than a'_{i-1}, $a_{i-1} = b + \cdots$ and $a_i = b - \cdots$ and so the "$-$" after b must not be discarded and we get a contradiction. Then, at least a_i must be written $a_i = a'_{i-1}(-)^{m_i+1}\cdots$. The last "$-$" may be discarded if and only if $a_{i-1} = a'_{i-1}(-)^{m_i}$ and r_{i-1} is not dyadic, in the case $n_i = m_i + 1$. Otherwise it is the first term of a_{i_1} and $n_i = m_i$. □

Lemma 3.3.33. *Let $(a_i)_i$ be a decreasing sequence of surreal numbers and $(r_i)_i$ be real numbers. Let i be a limit ordinal. Let $E = \{\alpha \mid \exists j < i \quad \forall \beta \leq \alpha \forall j < k < i \quad a_k(\beta) = a_j(\beta)\}$. Then E is a transitive set of ordinals. For $\alpha \in E$ denote j_α the least ordinal that proves $\alpha \in E$. Let $a''_{i,0}(\alpha) = a_{j_\alpha}(\alpha)$ for $\alpha < \sup E$. We have to cases:*

- *$a''_{i,0}$ has a prefix such that its "$+$"s have order type l_i. Then $a_{i,0}$ is the longest such a prefix.*
- *Otherwise, $a''_{i,0}$ is a prefix of $a_{i,0}$ and $a_{i,0} = a''_{i,0}(+)^{l'_i}$ with l'_i such that the order type of $a''_{i,0}(+)^{l'_i}$ is l_i.*

Proof. E is obviously transitive, if j proves $\alpha \in E$ then for $\beta < \alpha$, then j also proves $\alpha \in E$.

- Assume $a''_{i,0}$ has a prefix such that its "$+$"s have order type l_i. Let u be such a prefix that maximizes the length. Let v the longest common prefix to u and $a_{i,0}$. If it is shorter and $a_{i,0}$ then, $a_{i,0} = v(+)^k$ for some ordinal $k > 0$. Indeed every "$-$" in $a_{i,0}$ must be discarded and a "$-$" after v cannot be discarded by any a_j. Moreover the "$+$"s of v cannot have order type l_i since $k > 0$ then $u = v - \cdots$ that is $a_{i,0} > a_j$ for some $j < i$ and then $a_i > a_j$ for such a j, what is impossible. Then $v = a_{i,0}$. That is $a_{i,0}$ is a prefix of u. Since there "$+$"s have same order type there is some ordinal k such that $u = a_{i,0}(-)^k$. Assume $k > 0$. Then $a_{i,1}$ must not be empty otherwise for some $j < i$ we would have $a_j < a_i$. Then $a_{i,1}$ starts with a "$-$" that will be discarded what is again a contradiction. Then $u = a_{i,0}$.
- Assume $a''_{i,0}$ does not have such a prefix. Let u the longest common prefix of $a_{i,0}$ and $a''_{i,0}$. If it is shorter that $a''_{i,0}$, then $a''_{i,0} = u + \cdots$ and $a_{i,0} = u - \cdots$ ($a_{i,0}$ is not finished because the order type of the "$+$"s of i is less than l_i), otherwise there is some $j < i$ such that $a_j < a_i$. But then the first "$-$" in $a_{i,0}$ after u cannot be discarded. What is a contradiction with the definition of $a_{i,0}$. Then $u = a''_{i,0}$. Since we cannot discard any "$-$" further we have $a_{i,0} = a''_{i,0}(+)^{l'_i}$ for some ordinal l'_i.

□

Proof of correctness of Algorithm 1. Corollaries 3.3.30 and 3.3.31 identifies where we have to find the discarded minuses. Lemma 3.3.33 ensure that the case when A has length a limit ordinal is correct and Lemma 3.3.32 provides the successor ordinal case. □

Then, the following algorithm performs the computation of the series from the signs sequence:

Algorithm 2 Computing the series

```
Input : s a sign expansion
α ← 0                          ▷ counter for position in the sign expansion
β ← 0                          ▷ counter for the position in the serie
A, R ← [], []                  ▷ Initialize the list of exponent and coefficients to the empty list
loop
    nb_+ ← 0
    a ← ()                     ▷ Initialize a to the empty string
    r ← +                      ▷ By default r is positive
    p, m ← +, −
    if s(α) = − then           ▷ If we are looking at an ω^a r with r < 0, we switch the roles of + and −
        p, m = −, +
        r ← −                  ▷ We also know that r starts with a −
    end if
    Identifying ω^{a°} :
    loop
        if p^{nb_+ +1} is a prefix of s[α :] then
            a ← a+
            α ← α + nb_+ + 1
            nb_+ ← nb_+ + 1
        else
            if m^{nb_+ +1} is a prefix of s[α :] then
                a ← a−
                α ← α + nb_+ + 1
            else
                Break
            end if
        end if
    end loop                   ▷ At the end of this loop we have identified ω^{a°}
    Identifying r :
    loop
        if p^{nb_+} is a prefix of s[α :] then
            r ← rp             ▷ Concatenation of strings
            α ← α + nb_+
        else
            if m^{nb_+ +1} is a prefix of s[α :] then
                r ← rm
                α ← α + nb_+
            else
                Break
            end if
        end if
    end loop
    a ← unReduce(a, A, R)
    (A, R) ← (A.a, R.r)
end loop
return (A, R)
```

To conclude this part, we state some lemmas about the length of the signs sequences and the length of the surreal numbers involved in the normal form.

Lemma 3.3.34 ([48, van den Dries and Ehrlich, Lemma 4.1]). *For all surreal number $a \in$ **No**,*

$$|a|_{+-} \leq |\omega^a|_{+-} \leq \omega^{|a|_{+-}}$$

Lemma 3.3.35 ([26, Gonshor, Lemma 6.3] and [48, van den Dries and Ehrlich, Lemma 4.2]). *Let $x = \sum_{i<\nu} r_i\omega^{a_i}$ a surreal number. We have:*

- $\nu \leq |x|_{+-}$
- *for all $i < \nu$, $|r_i\omega^{a_i}|_{+-} \leq |x|_{+-}$*
- *if there is some α such that or all $i < \nu$, $|r_i\omega^{a_i}|_{+-} \leq \alpha$, then $|\leq|_{+-} \alpha\nu$.*

3.4 Substructures stable under the fields operations

Since **No** is a very large field (it is not even a set) it would be nice to exhibit some interesting subfield of it, or at least some algebraic substructure. This work as already been done in the literature, in particular in [**?**]. As an simple example, we may wonder what happens when we bound the birthday (or a length) of surreal numbers.

Theorem 3.4.1 ([48, 47, van den Dries and Ehrlich, Corollaries 3.1, 4.4 and 4.9])**.** *The ordinals λ such that $\mathbf{No}_\lambda$ is closed under the various fields operations of* **No** *can be characterized as follows:*

- $\mathbf{No}_\lambda$ *is an additive subgroup of* **No** *iff $\lambda = \omega^\alpha$ for some ordinal α, i.e. λ is an additive ordinal number.*
- $\mathbf{No}_\lambda$ *is a subring of* **No** *iff $\lambda = \omega^{\omega^\alpha}$ for some ordinal α, i.e. λ is a multiplicative ordinal number.*
- $\mathbf{No}_\lambda$ *is a subfield of* **No** *iff $\omega^\lambda = \lambda$, i.e. λ is an ε-number.*

The proof of the previous theorem relies on some useful lemmas recalled below:

Lemma 3.4.2 ([26, Gonshor, Theorem 6.1])**.** *For all surreal numbers a and b, $|a+b|_{+-} \leq |a|_{+-} + |b|_{+-}$.*

Remark 3.4.3. The sum of ordinal numbers in the above theorem is the natural sum (i.e. the Hessenberg sum, the sum of ordinal seen as surreal numbers).

We have something similar for multiplication.

Lemma 3.4.4 ([**?**, van den Dries and Ehrlich, Corollary 4.3])**.** *For all surreal numbers a and b, $|ab|_{+-} \leq \omega\, |a|_{+-}^2\, |b|_{+-}^2$. If there is $x \in \mathbf{No}$ and $r \in \mathbb{R}$ squch that $b = \omega^x s$ we can even do better: $|ab|_{+-} \leq |a|_{+-}^2\, |b|_{+-}$.*

The previous bound was shown as a large improvement of Gonshor's bound $|ab|_{+-} \leq 3^{|a|_{+-}+|b|_{+-}}$. Gonshor also conjectured that $|ab|_{+-} \leq |a|_{+-}\, |b|_{+-}$ but such a bound as not been achieved yet.
The last point of Theorem 3.4.1 is a consequence of strong decomposition theorem which is state above and a lemma which is the following:

Lemma 3.4.5 ([**?**, **?**, van den Dries and Ehrlich, Lemma 4.8])**.** *If α is an ordinal number but not an ε-number, then $|\omega^{-\alpha}|_{+-} < \omega^\alpha$.*

Theorem 3.4.6 ([**?**, **?**, van den Dries and Ehrlich, Proposition 4.7])**.** *Let λ be an ε-number. Then*

1. $\mathbf{No}_\lambda$ *can be expressed as*
$$\mathbf{No}_\lambda = \bigcup_\mu \mathbb{R}_\lambda^{\mathbf{No}_\mu}, \tag{3.9}$$
where μ ranges over the additive ordinals less than λ (equivalently, μ ranges over the multiplicative ordinals less than λ).
2. $\mathbf{No}_\lambda$ *is a real closed subfield of* **No***, and is closed under the restricted analytic functions of* **No***.*
3. $\mathbf{No}_\lambda = \mathbb{R}_\lambda^{\mathbf{No}_\lambda}$ *if and only if λ is a regular cardinal.*

This theorem gives a decomposition of $\mathbf{No}_\lambda$ into Hahn fields. We will actually be very interested in fields of the form $\mathbb{R}_\lambda^{\mathbf{No}_\mu}$ in the following of this thesis.

3.5 Surreal substructures

In this section we give an introduction to surreal substructure as it is an important notion to understand what the λ-numbers and the κ-numbers are (see Sections 3.7.2 and 3.7.3). This section is mostly based on the work of Bagayoko and van der Hoeven [9].

Definition 3.5.1 (Surreal substructure, [9, Bagayoko and van der Hoeven, Definition 4.1])**.** Let $S \subseteq \mathbf{No}$ be a subclass of **No**. S is called a **surreal substructure** if there is a surjective function $f : \mathbf{No} \to S$ that is increasing for both $\leq$ and $\sqsubseteq$:

$$x \sqsubset y \implies f(x) \sqsubset f(y) \qquad \text{and} \qquad x < y \implies f(x) < f(y)$$

In other word f is an homomorphism of partially ordered set for both $\leq$ and $\sqsubseteq$.

Example 3.5.2.

- $\{x \in \mathbf{No} \mid x > 0\}$ is a surreal substructure. The homomorphism consist in, given x, writing $+$ and then the signs sequence of x. More generally, starting with any fixed signs expansion and then writing the signs expansion of the input is a surreal substructure.

- If S is a substructure with the homomorphism f, $-S$ is a substructure with the homomorphism $-f \circ (-\operatorname{id})$.

Proposition 3.5.3 ([9, Bagayoko and van der Hoeven, Proposition 4.9]). *Let S be a surreal substructure. Then there is a unique function $\Xi_S : \mathbf{No} \to S$ that is a homomorphism of partially ordered set for both $\leq$ and $\sqsubseteq$.*

Proof. The existence is given by the definition. Assume there two such homomorphisms f and g.

- Since f and g are increasing for $\sqsubset$ then $f(0) = g(0) = s$ where s is the simpler element of S.
- Assume that for all y such that $|y|_{+-} < |x|_{+-}$, $f(y) = g(y)$. Write $x = [L \mid R]$ in canonical form. Since f and g are increasing for $\leq$ and $\sqsubseteq$, both $f(x)$ and $g(x)$ are the simplest element of S that are greater than $f(L) = f(L)$ and less than $f(R) = g(R)$. Therefore, $f(x) = g(x)$.

Hence, by transfinite induction, $f = g$. □

3.6 Gonshor's exponential and logarithm

3.6.1 Gonshor's exponential

In this section, we present the function exp over the surreal numbers. We waste no time to give its definition:

Definition 3.6.1 (Function exp, [26, page 145]). Let $x = [x' \mid x'']$ be the canonical representation of x. We define inductively

$$\exp x = \left[0, \exp(x')[x - x']_n, \exp(x'')[x - x'']_{2n+1} \;\middle|\; \frac{\exp(x')}{[x' - x]_{2n+1}}, \frac{\exp(x'')}{[x'' - x]_n} \right]$$

where n ranges in $\mathbb{N}$ and where

$$[x]_n = 1 + \frac{x}{1!} + \cdots + \frac{x}{n!},$$

with the further convention that the expressions containing terms of the form $[y]_{2n+1}$ are to be considered only when $[y]_{2n+1} > 0$.

As Gonshor showed in [26, Theorem 10.1], this is well defined and exp is an increasing positive function.

Proposition 3.6.2 ([26, Gonshor, Corollary 10.1]). exp *has the uniformity property.*

For appreciable number, the value of exp is exactly what we would expect from the series.

Theorem 3.6.3 ([26, Theorems 10.2, 10.3 and 10.4]). *For all $r \in \mathbb{R}$ and ε infinitesimal, we have*

$$\exp r = \sum_{k=0}^{\infty} \frac{r^k}{k!} \quad \text{and} \quad \exp \varepsilon = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \quad \text{and} \quad \exp(r+\varepsilon) = \exp(r)\exp(\varepsilon) = \sum_{k=0}^{\infty} \frac{(r+\varepsilon)^k}{k!}$$

Moreover for all purely infinite number x,

$$\exp(x + r + \varepsilon) = \exp(x)\exp(r + \varepsilon)$$

This fact mostly comes from the fact that we use the asymptotic development of $\exp x$ in x' and x'' and that appreciable numbers are always at an appreciable distance of each of their prefixes, thus being the the convergence radius of the series (which is infinite). This fact is not true for infinite numbers. However we can study in detail the behavior of exp with purely infinite numbers so that, using the previous theorem, we have control on what happens for any surreal number.

Proposition 3.6.4 ([26, Theorem 10.7]). *If x is purely infinite, then $\exp x = \omega^a$ for some surreal number a.*

More precisely:

Proposition 3.6.5 (Function g, [26, Theorem 10.13]). *If x is purely infinite, i.e. $x = \sum_{i<\nu} r_i \omega^{a_i}$ with $a_i > 0$ for all i, then*

$$\exp x = \omega^{\sum_{i<\nu} r_i \omega^{g(a_i)}},$$

for some function $g : \mathbf{No}_+^* \to \mathbf{No}$ which satisfies for all x,

$$g(x) = [c(x), g(x') \mid g(x'')]$$

where $c(x)$ is the unique number such that $\omega^{c(x)} \asymp x$ and where x' ranges over the lower non-zero prefixes of x and x'' over the upper prefixes of x.

The function g may look like the identity but is quite more complex. For instance, it consists in adding 1 for ordinals which are "close" to an ε-numbers

Proposition 3.6.6 ([26, Theorem 10.14]). *If a is an ordinal number then*

$$g(a) = \begin{cases} a+1 & \text{if } \lambda \leq a < \lambda + \omega \text{ for some } \varepsilon\text{-number } \lambda \\ a & \text{otherwise} \end{cases}$$

Note that in the previous proposition, $a \neq 0$ since g is defined only for positive elements.
For infinitely small elements, g ranges over all the negative surreal numbers. For instance:

Proposition 3.6.7 ([26, Theorem 10.15]). *Let n be a natural number and b be an ordinal. We have $g(2^{-n}\omega^{-b}) = -b+2^{-n}$.*

Proposition 3.6.8 ([26, Theorems 10.17, 10.19 and 10.20]). *If b is a surreal number such that for some ε-number ε_i, some ordinal α and for all natural number n, $\varepsilon_i + n < b < \alpha < \varepsilon_{i+1}$, then $g(b) = b$. This is also true if there is some ordinal $\alpha < \varepsilon_0$ such that for all natural number b, $n\omega^{-1} < b < \alpha < \varepsilon_0$.*

Proposition 3.6.9 ([26, Theorem 10.18]). *If $\varepsilon \leq b \leq \varepsilon + n$ for some ε-number ε and some integer n. In particular, the sign expansion of b is the signs sequence of ε followed by some signs sequence S. Then, the signs sequence of $g(b)$ is the signs sequence of ε followed by a $+$ and then S. In particular, $g(b) = b+1$.*

Remark 3.6.10. Note that Propositions 3.6.8 and 3.6.9 do not handle the case where $x = \varepsilon_\alpha - y$ with $y > 0$ and $y \prec \varepsilon_{\alpha+1}$.

Notation. We will denote for all $n \in \mathbb{N}$ $\exp_n$ the n-fold composition of $\exp$. $\exp_0$ is the identity, $\exp_1$ is $\exp$, $\exp_2$ is $\exp \circ \exp$ and so on.

3.6.2 Gonshor's logarithm

Gonshor has shown that we can define the logarithm over positive surreal numbers. The definition goes first on monomials ω^a for some surreal a and for appreciable numbers. Indeed, Proposition 3.6.5 has shown that the exponential of purely infinite number are of the form ω^a and Theorem 3.6.3 has shown that appreciable numbers are sent to appreciable numbers. We then need to to the other direction. Following Gonshor, we introduce:

Definition 3.6.11 ([26, Gonshor]). For a surreal number a in canonical representation $a = [a' \mid a'']$, we define

$$\ln \omega^a = \left[\begin{array}{c|c} \left\{ \ln \omega^{a'} + n \;\middle|\; \begin{array}{c} n \in \mathbb{N} \\ a' \sqsubset a \\ a' < a \end{array} \right\} & \left\{ \ln \omega^{a''} - n \;\middle|\; \begin{array}{c} n \in \mathbb{N} \\ a'' \sqsubset a \\ a < a'' \end{array} \right\} \\ \left\{ \ln \omega^{a''} - \omega^{\frac{a''-a}{n}} \;\middle|\; \begin{array}{c} n \in \mathbb{N} \\ a'' \sqsubset a \\ a < a'' \end{array} \right\} & \left\{ \ln \omega^{a'} + \omega^{\frac{a-a'}{n}} \;\middle|\; \begin{array}{c} n \in \mathbb{N} \\ a' \sqsubset a \\ a' < a \end{array} \right\} \end{array} \right]$$

Theorem 3.6.12 ([26, Gonshor, Theorem 10.8]). *$\ln \omega^a$ is well defined for all surreal number a and for all surreal numbers a, b such that $a > b$, and for all $n \in \mathbb{N}$,*

$$0 < \ln \omega^a - \ln \omega^b < \omega^{\frac{a-b}{n}}$$

in particular, if $a > 0$, then for all $n \in \mathbb{N}^*$,

$$0 < \ln \omega^a < \omega^{\frac{a}{n}}$$

As often we have:

Lemma 3.6.13 ([26, Gonshor, Lemma 10.1]). *The definition of $\ln \omega^a$ has the uniform property.*

As expected, $\ln$ is the compositional inverse of $\exp$ (for monomial at least):

Proposition 3.6.14 ([26, Theorem 10.8]). *For all surreal number a, $\ln \omega^a$ is purely infinite.*

Theorem 3.6.15 ([26, Theorem 10.9]). *For all surreal number a, $\exp \ln \omega^a = \omega^a$.*

In Proposition 3.6.5, we got a nice formula for the exponential of a purely infinite surreal number. Now it is time to get the other direction. We had to use a function g in Proposition 3.6.5; This time, we will have to use a new function which is denoted h by Gonshor.

Definition 3.6.16.

$$h(b) = \left[0, h(b') \;\middle|\; h(b''), \frac{\omega^b}{n} \right]$$

This expression is uniform (see [26]) and then does not depend of the expression of b as $[b' \mid b'']$.

Theorem 3.6.17 ([26, Theorem 10.12]). *For all surreal number a, $\ln \omega^{\omega^a} = \omega^{h(a)}$.*

The above theorem is not actually stated like this in [26] but this statement follows from the proof Gonshor came up with.
As a consequence of Theorems 3.6.15 and 3.6.17 and Propositions 3.6.14 and 3.6.5, we have

Corollary 3.6.18. *For all surreal number $a = \sum_{i<\nu} r_i \omega^{a_i}$, we have*

$$\ln \omega^a = \sum_{i<\nu} r_i \omega^{h(a_i)}$$

Finally, since for appreciable numbers exp is defined by its usual series, $\ln(1+x)$ is also defined by its usual series when x in infinitesimal. More precisely,

Definition 3.6.19. For x an infinitesimal,

$$\ln(1+x) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1} x^i}{i}$$

As a consequence of Theorem 3.6.3, we end up with

Corollary 3.6.20. *Let $a = \sum_{i<\nu} r_i \omega^{a_i}$ a positive surreal number. Then*

$$\ln a = \ln \omega^{a_0} + \ln r_0 + \ln \left(1 + \sum_{1 \leq i < \nu} \frac{r_i}{r_0} \omega^{a_i - a_0}\right)$$

where the last term is defined in Definition 3.6.19.

Notation. As for the exponential function, we will denote for all $n \in \mathbb{N}$ $\ln_n$ the n-fold composition of ln. $\ln_0$ is the identity, $\ln_1$ is ln, $\ln_2$ is $\ln \circ \ln$ and so on.

3.6.3 More about functions g and h

It is possible to bound the length of $g(a)$ depending on the length of a.

Lemma 3.6.21 ([48, Lemma 5.1]). *For all $a \in \mathbf{No}$, $|g(a)|_{+-} \leq |a|_{+-} + 1$.*

Corollary 3.6.22. *If a is an ordinal number then $h(-a) = \omega^{-a-1}$.*

Proof. It is a direct consequence of Proposition 3.6.7 and the fact that $h = g^{-1}$. ☐

As for g, we can bound the length of $h(a)$ in function of the length of a.

Lemma 3.6.23 ([6, Proposition 3.1]). *For all $a \in \mathbf{No}$, $|h(a)|_{+-} \leq \omega^{|a|_{+-}+1}$*

We will also prove another lemma, Lemma 3.6.25, that looks like the previous lemma but that is better in many cases but not always. To do so we first prove another technical lemma.

Lemma 3.6.24. *Let c be a surreal number. Assume $g(a) < c$ for all $a \sqsubset \omega^c$ such that $0 < a < \omega^c$. Then $g(\omega^c) = c_+$ if c does not have a longest prefix greater than itself, otherwise, $g(\omega^c) = c''$ where c'' is the longest prefix of c such that $c'' > c$.*

Proof. By induction on c:

- For $c = 0$, $g(\omega^0) = g(1) = 1$ whose signs sequence is indeed the one of 0 followed by a plus.
- Assume the property for $b \sqsubset c$. Assume $g(a') < c$ for all $a' \sqsubset \omega^c$ such that $0 < a' < \omega^c$. Then,

  $$g(\omega^c) = [c \mid g(a'')]$$

  where a'' ranges over the elements such that $a'' \sqsubset \omega^c$ and $a'' > \omega^c$.

  ➢ First case: c has a longest prefix c_0 such that $c_0 > c$. Let a'' such that $a'' \sqsubset \omega^c$ and $a'' > \omega^c$. Let c'' such that $a'' \asymp \omega^{c''}$. We necessarily have $c'' \sqsubset c$ and $c'' > c$. Therefore, $c'' \sqsubset c_0$ and $c'' > c_0$. Thus

  $$c < c_0 < c'' < g(a'')$$

  By the simplicity property ensures $g(\omega^c)$ is a prefix of all surreal x such that $c < x < g(a'')$ for all a'' such that $a'' \sqsubset \omega^c$ and $a'' > \omega^c$. Hence,

  $$g(\omega^c) \sqsubseteq c_0 \sqsubset c$$

  For any $a < \omega^{c_0}$ such that $a \sqsubset \omega^{c_0}$, we also have $a \sqsubset \omega^c$ and $a < \omega^c$. Thus $g(a) < c < c_0$. We then can apply the induction hypothesis on ω^{c_0} and get one of the following cases

∵ If c_0 as a longest prefix c_1 such that $c_1 > c_0$, then $g(\omega^{c_0}) = c_1$. In terms of signs sequences, we have some ordinals α and β such that

$$c_0 = (c_1)(-)(+)^\alpha \qquad \text{and} \qquad c = (c_0)(-)(+)^\beta$$

Thus
$$c_0 = [c \mid c_1]$$
But, by definition $c < g(\omega^c) < g(\omega^{c_0}) = c_1$. Thus, by the simplicity property, $c_0 \sqsubseteq g(\omega^c)$. Finally, $g(\omega^c) = c_0$.

∵ If c_0 has no longest prefix c_1 such that $c_1 > c_0$, then $g(\omega^{c_0}) = (c_0)_+$. In terms of signs sequences, we have some ordinal α such that

$$c_0 = (c_1)(-)(+)^\alpha$$

Thus
$$c_0 = \left[c \,\middle|\, (c_0)_+\right]$$
But, by definition $c < g(\omega^c) < g(\omega^{c_0}) = (c_0)_+$. Thus, by the simplicity property, $c_0 \sqsubseteq g(\omega^c)$. Finally, $g(\omega^c) = c_0$.

➢ Second case: c does not have a longest prefix greater than c. Then,

$$g(\omega^c) = \left[c \,\middle|\, g(\omega^{c''})\right]$$

where c'' ranges over the prefixes of c greater than c. Let $d \sqsubset c$ such that $d > c$. Then there is d_1 of minimal length such that $d \sqsubset d_1 \sqsubset c$ and $d_1 > c$. By minimality of d_1, d is the longest prefix of d_1 greater than d_1. As in the first case, we can apply the induction hypothesis on d_1 and get $g(\omega^{d_1}) = d$. Thus all the prefixes c greater than c appear in the elements $g(\omega^{c''})$ for c'' a prefix of c greater than c. The only other possible value of $g(\omega^{c''})$ is d_+ for some d a prefix of c greater than c. Hence it has no effect in the computation of $g(\omega^c)$. Finally,

$$g(\omega^c) = \left[c \,\middle|\, c'', c''_+\right] = [c \mid c'']$$

where c'' ranges over the prefixes of c greater than c. We finally conclude that $g(\omega^c) = c_+$.

□

We recall that we denote $\oplus$ the usual addition (see Definition 2.3.1) over the ordinal numbers and $\otimes$ the usual product (see Definition 2.3.4) over ordinal numbers. They are not the addition and multiplication of ordinal number seen as surreal numbers.

Lemma 3.6.25. *For all $a > 0$, $|a|_{+-} \leq \left|\omega^{g(a)}\right|_{+-} \otimes (\omega + 1)$.*

Proof. We proceed by induction on $|a|_{+-}$.

- For $a = 1$, $g(a) = 1$ and we indeed have $1 \leq \omega^2 + \omega$.
- Assume the property for all $b \sqsubset a$. Let c such that $\omega^c \asymp a$. Then

$$g(a) = [c, g(a') \mid g(a'')]$$

We split into two cases:

➢ If there is some $a_0 \sqsubset a$ such that $a_0 < a$ and $g(a_0) \geq c$ then

$$g(a) = [g(a') \mid g(a'')]$$

Let S the signs sequence such that the signs sequence of a is the signs sequence of a_0 followed by S. By an easy induction on the length of S, we can show that the signs sequence of $g(a)$ is the signs sequence of $g(a_0)$ followed S. Let α the length of S. Therefore using Theorem 3.3.28,

$$\left|\omega^{g(a)}\right|_{+-} \geq \left|\omega^{g(a_0)}\right|_{+-} \oplus (\omega \otimes \alpha)$$

and then,

$$\begin{aligned}
\left|\omega^{g(a)}\right|_{+-} \otimes (\omega + 1) &\geq \left(\left|\omega^{g(a_0)}\right|_{+-} \oplus (\omega \otimes \alpha)\right) \otimes (\omega + 1) \\
&\geq \left(\left|\omega^{g(a_0)}\right|_{+-} \oplus (\omega \otimes \alpha)\right) \otimes \omega \oplus \left|\omega^{g(a_0)}\right|_{+-} \oplus (\omega \otimes \alpha) \\
&\geq \left|\omega^{g(a_0)}\right|_{+-} \otimes \omega \oplus \left|\omega^{g(a_0)}\right|_{+-} \oplus (\omega \otimes \alpha) \\
&\geq \left|\omega^{g(a_0)}\right|_{+-} \otimes (\omega + 1) \oplus (\omega \otimes \alpha)
\end{aligned}$$

and by induction hypothesis on a_0,

$$\left|\omega^{g(a)}\right|_{+-} \otimes (\omega+1) \geq |a_0|_{+-} \oplus (\omega \otimes \alpha) \geq |a_0|_{+-} \oplus \alpha = |a|_{+-}$$

➢ Otherwise, for any $a_0 \sqsubset a$ such that $a_0 < a$, $g(a_0) < c$. Therefore,

$$g(a) = [c \mid g(a'')]$$

Also, since $a > 0$, we can write the signs sequence of a as the one of ω^c followed by some signs sequence S. If S contains a plus, then there is a prefix of a, a_0 such that $a_0 < a$ and still $a_0 \asymp \omega^c$ and then $g(a_0) > c$ what is not the case by assumption. Then, S is a sequence of minuses. Let α be the length of S. Again, by an easy induction on α, the signs sequence of $g(a)$ is the one of $g(\omega^c)$ followed by S. Hence,

$$\left|\omega^{g(a)}\right|_{+-} \geq \left|\omega^{g(\omega^c)}\right|_{+-} \oplus (\omega \otimes \alpha)$$

If S is not the empty signs sequence, as in the previous case but using the induction hypothesis on ω^c,

$$\left|\omega^{g(a)}\right|_{+-} \otimes (\omega+1) \geq |\omega^c|_{+-} \oplus \alpha = |a|_{+-}$$

Now if S is the empty sequence, $a = \omega^c$. Applying Lemma 3.6.24 to c we get that either $g(a) = c_+$ or $g(a)$ is the last prefix of c greater than c. If the first case occurs then a is a prefix of $\omega^{g(a)}$ and then $\left|\omega^{g(a)}\right|_{+-} \geq |a|_{+-}$. Now assume that the second case occurs. Then for any b such that $g(a) \sqsubset b \sqsubset c$, $b < c$. If for all $b' \sqsubset b$ such that $b' < b$, $g(b') < b$, then Lemma 3.6.24 applies. Since b has a last prefix greater than itself, $g(a)$, $g(\omega^b) = g(a)$ and we reach a contradiction since $b < c$ and therefore $\omega^b < \omega^c = a$. Then for all b such that $g(a) \sqsubset b \sqsubset c$, there is some $b' \sqsubset b$, $b' < b$ such that $g(\omega^{b'}) > b$. Since the signs sequence of b consists in the one of $g(a)$ a minus and then a bunch of pluses, and since $g(\omega^{b'})$ must also a a prefix of c, $g(\omega^{b'}) \sqsubseteq g(a) \sqsubset b$. Therefore to ensure $g(b') > b$, we must have $g(\omega^{b'}) \geq g(a)$. Since $\omega^{b'}$ is a prefix of a lower than a, it is a contradiction. Therefore, there is no b such that $g(a) \sqsubset b \sqsubset c$ and $b < c$, and finally, the signs sequence of c is the one $g(a)$ followed by a minus. In particular, $g(a)$ and c have the same amount of pluses, say α. Then, using Theorem 3.3.28,

$$\begin{aligned}
|a|_{+-} &= \left|\omega^{g(a)}\right|_{+-} \oplus \omega^{\alpha+1} \\
&\leq \left|\omega^{g(a)}\right|_{+-} \oplus \left|\omega^{g(a)}\right|_{+-} \otimes \omega = \left|\omega^{g(a)}\right|_{+-} \otimes \omega \\
&\leq \left|\omega^{g(a)}\right|_{+-} \otimes (\omega+1)
\end{aligned}$$

We conclude using the induction principle.

□

Corollary 3.6.26. *For all $a > 0$ and for all multiplicative ordinal greater than ω, if $|a|_{+-} \geq \mu$, then $\left|\omega^{g(a)}\right|_{+-} \geq \mu$.*

Proof. Assume the that $\left|\omega^{g(a)}\right|_{+-} < \mu$. Then using Lemma 3.6.25, $\mu \leq \left|\omega^{g(a)}\right|_{+-} \otimes (\omega+1)$. Since μ is a multiplicative ordinal greater than ω, we have $\omega + 1 < \mu$. μ is a multiplicative ordinal, hence $\left|\omega^{g(a)}\right|_{+-} \otimes (\omega+1) < \mu$ and we reach a contradiction. □

3.7 Log-atomic numbers

3.7.1 Generalities

We now introduce the concept of log-atomic numbers. Log-atomic numbers were first introduced by Schmeling in [41, page 30] about transseries. A log-atomic number is a number that stays simple when taking its logarithm. Namely, when looking at the series writing of a surreal number or to a transseries, taking the logarithm does not explode the number of terms, namely, it remains equal to 1.

Definition 3.7.1 (Log-atomic)**.** A positive surreal number $x \in \mathbf{No}^*_+$ is said **log-atomic** *iff* for all $n \in \mathbb{N}$, there is a surreal number a_n such that $\ln_n x = \omega^{a_n}$. We denote $\mathbb{L}$ the class of log-atomic numbers.

Remark 3.7.2. Actually, all the a_ns can be taken positive since if $a_n \leq 0$ then $\ln_{n+1} x \leq 0$ which is impossible because of the existence of a_{n+1}.

Example 3.7.3. A typical example is ω. We can check that for all $n \in \mathbb{N}$, $\ln_n \omega = \omega^{\frac{1}{\omega^n}}$. Therefore

$$\{\exp_n \omega, \ln_n \omega \mid n \in \mathbb{N}\} \subseteq \mathbb{L}$$

In transseries fields (see Section 4.2 for definitions and details), log-atomic numbers are the number that we choose to be the fundamental bricks to build numbers. In surreal number it is the very same thing. Log-atomic number are the number we cannot divide into simpler numbers when considering exponential and logarithm and are the fundamental bricks we end up with when writing $x = \sum_{i<\nu} r_i \omega^{a_i}$ and then each ω^{a_i} as $\omega^{a_i} = \exp x_i$ with x_i a purely infinite number and then doing the same thing with each of the x_is. The use of the word "simpler" is not innocent. Indeed, log-atomic numbers are also the simplest elements for some equivalence relation as the ω^as are the simplest elements in the classes of the equivalence relation $\asymp$.

Definition 3.7.4 ([12, Berarducci and Mantova, Definition 5.2])**.** Let x, y be two positive infinite surreal numbers. We write

- $x \asymp^L y$ *iff* there are some natural numbers n, k such that $\exp_n\left(\frac{1}{k}\ln_n y\right) \leq x \leq \exp_n(k \ln_n y)$. Equivalently, we ask that the is a natural number n such that $\ln_n x \asymp \ln_n y$. For such n we notice that $\ln_{n+1} x \sim \ln_{n+1} y$.
- $x \prec^L y$ *iff* for all natural numbers n and k, $x < \exp_n\left(\frac{1}{k}\ln_n y\right)$. Equivalently, we ask that for all $n \in \mathbb{N}$, $\ln_n x \prec \ln_n y$.
- $a \preceq^L b$ *iff* there are some natural numbers n and k, $x \leq \exp_n\left(\frac{1}{k}\ln_n y\right)$. Equivalently, we ask that for some $n \in \mathbb{N}$, $\ln_n x \preceq \ln_n y$.

It is common exercise to check that $\asymp^L$ is an equivalence relation and that $\preceq^L$ is a preorder which is associated with $\asymp^L$. We claimed that log-atomic numbers are the simplest and each of the equivalence classes of $\asymp^L$. First, we can notice that they are all in distinct classes.

Proposition 3.7.5 ([12, Berarducci and Mantova, Proposition 5.6])**.** *Let $x, y \in \mathbb{L}$ such that $x < y$. Then $x \prec^L y$.*

Proof. Assume $y \preceq^L x$. Then, by definition there is some n such that $\ln_n y \preceq \ln_n x$. If $\ln_n y \prec \ln_n x$ and then $\ln_n y < \ln_n x$ since both of them are monomials (see Definition 3.3.15). Then, the fact that exp is increasing ensure that $y < x$, which is a contradiction. Then, $\ln_n x \asymp \ln_n y$. Again, since they are monomials, $\ln_n x = \ln_n y$ and then $x = y$ which is again a contradiction. □

We now prove that in each class there is a log-atomic number and that it is the simplest number in its class.

Proposition 3.7.6 ([12, Berarducci and Mantova, Proposition 5.8])**.** *For all positive infinite x there is some log-atomic number $y \in \mathbb{L}$ such that $y \sqsubseteq x$ and such that $y \asymp^L x$.*

The following proof is exactly the proof given by Berarducci and Mantova. But since the actual theorem was not stated exactly the same way as ours, basically because we introduce things in a different order, we give their proof translated with our words and notations.

Proof. ([12, Berarducci and Mantova]) Assume that the property is not true and that there is a counter example x. Therefore, no positive infinite prefix y of x is such that $y \in \mathbb{L}$ and $y \asymp^L x$. Without loss of generality, we also may assume that x has minimal length in its class: if $y \sqsubset x$, then $y \not\asymp^L x$.

$$x = x_0 = \sum_{i<\nu_0} r_{0,i} \exp(x_{0,i})$$

and more generally

$$x_{n,0} = x_{n+1} = \sum_{i<\nu_{n+1}} r_{n+1,i} \exp(x_{n+1,i})$$

with all the $x_{n,i}$s being positive purely infinite numbers. We prove by induction on n that $r_{n,0} = 1$ and that $\nu_n = 1$.

- For $n = 0$, if $\nu_0 > 1$ or if $r_{0,0} \neq 1$ then $r_{0,0}\exp(x_{0,0}) \asymp^L x$. This contradicts the minimality of x. Therefore $r_{0,0} = \nu_0 = 1$.
- Assume the property shown up to some natural number n. Then $x = \exp(\exp(\cdots \exp(x_{n+1}) \cdots))$. Therefore again, we have $y = \exp(\cdots \exp(\exp(x_{n+1,0})) \cdots) \asymp^L x$ and thanks to Proposition 3.6.5 and Theorem 3.3.28, we get that $y \sqsubseteq x$. By minimality of x, $x = y$, that is $\nu_{n+1} = r_{n+1,0} = 1$.

Using the induction principle, we conclude that x is a log-atomic number. In particular, it cannot be a counter-example. We reach a contradiction. □

As an immediate corollary, we have:

Corollary 3.7.7 ([12, Berarducci and Mantova, Corollary 5.9])**.** *The element of $\mathbb{L}$ are exactly the simplest elements in each equivalence class of $\asymp^L$.*

3.7.2 The κ-map

In this section, we introduce the κ-map, $x \mapsto \kappa_x$. This map was originally introduced by Kuhlmann and Matusinski in [33]. Numbers of the form κ_x will be called κ-numbers. It was the first attempt to give a quasi-complete parametrization of the log-atomic numbers: They conjectured that all the log atomic number could be written as $\exp_n \kappa_x$ or $\ln_n \kappa_x$. It turned out that it was not true, as we will see in the next subsection.

Definition 3.7.8 ([33, Kuhlmann and Matusinski, Definition 3.1], [12, Berarducci and Mantova, Definition 5.19])**.** Let x, y be two positive infinite surreal numbers. We write

- $x \preceq^K y$ *iff* there are some natural number n such that $x \leq \exp_n y$.
- $x \prec^K y$ *iff* for all natural number n, $x < \ln_n y$.
- $a \asymp^K b$ *iff* there is some natural numbers n such that $\ln_n y \leq x \leq \exp_n y$.

Again, $\asymp^K$ is an equivalence relation, $\preceq^K$ is a preorder and $\prec^K$ the strict associated preorder. This relation is about exponential classes and is much easier to understand than the relation $\asymp^L$ and the corresponding preorders. The fact that it easy to understand leads to a the natural definition of the κ-map as follows:

Definition 3.7.9 ([33, Kuhlmann and Matusinski, Definition 3.1])**.** Let x be a surreal number and write it in canonical representation as $x = [x' \mid x'']$. Then we define

$$\kappa_x = [\mathbb{R}, \exp_n \kappa_{x'} \mid \ln_n \kappa_{x''}]$$

It is quite easy to see that $\kappa_0 = \omega$, $\kappa_{-1} = \omega^{\omega^{-\omega}}$ and $\kappa_1 = \varepsilon_0$. The κ-numbers are exactly the simplest elements in their respective classes for the equivalence relation $\asymp^K$. Indeed,

Proposition 3.7.10 ([33, Kuhlmann and Matusinski, Definition 3.1, Theorem 3.2])**.** *If $x < y$ then $\kappa_x \prec^K \kappa_y$. Moreover, the definition of κ_x has the uniformity property.*

Theorem 3.7.11 ([33, Kuhlmann and Matusinski, Definition 3.1, Theorem 3.4])**.** *A positive infinite surreal number is minimal in its equivalence class for $\asymp^K$ if and only if it is a κ-number.*

Proposition 3.7.12 ([12, Berarducci and Mantova, Proposition 5.3])**.** *If $x \asymp^L y$, then $x \asymp^K y$.*

With the previous fact we see that the conjecture of Kuhlmann and Matusinski is true if and only if $\left(\mathbf{No}_{>\mathbb{R}}/\asymp^K\right) \times \mathbb{Z}$ is order isomorphic to $\mathbf{No}_{>\mathbb{R}}/\asymp^L$. However, we will see that the ordered set $\mathbb{Z}$ must be replaced in the Cartesian product by some dense order, and $\mathbb{Z}$ is not a dense order. Therefore the conjecture is wrong. This fact is due to Berarducci and Mantova ([12]) using the λ-map.

Remark 3.7.13. The κ-map defines a surreal substructure.

3.7.3 The λ-map

Berarducci and Mantova have shown that the κ-maps fails to describe all the log-atomic numbers. However they provided an explicit description. The description basically uses the ideas of the definition of $\asymp^L$ in a similar way as the κ-map share the same principles as $\asymp^K$.

Definition 3.7.14 ([12, Berarducci and Mantova, Definition 5.12])**.** Let x be a surreal number and write it in canonical representation $x = [x' \mid x'']$. Then we define

$$\lambda_x = \left[\mathbb{R}, \exp_n\left(k \ln_n \lambda_{x'}\right) \,\middle|\, \exp_n\left(\frac{1}{k}\ln_n \lambda_{x''}\right)\right]$$

where $n, k \in \mathbb{N}^*$.

Proposition 3.7.15 ([12, Berarducci and Mantova, Proposition 5.13 and Corollary 5.15])**.** *The function $x \mapsto \lambda_x$ is well defined, increasing, satisfies the uniformity property and if $x < y$ then $\lambda_x \prec^L \lambda_y$.*

Proposition 3.7.16 ([12, Berarducci and Mantova, Proposition 5.16])**.** *For every $x \in \mathbf{No}$ with $x > \mathbb{R}$ there is a unique $y \in \mathbf{No}$ such that $x \asymp^L \lambda_y$ and $\lambda_y \sqsubseteq x$. In particular, λ_y is the simplest number in its equivalence class for $\asymp^L$.*

Proof. Assume that x has minimal length such that no element $y \in \mathbf{No}$ is such that $x \asymp^L \lambda_y$. In particular for every $z \sqsubset x$ such that $z > \mathbb{R}$, there is some y_z such that $\lambda_{y_z} \asymp^L z$ and therefore $z \not\asymp^L x$. As a consequence of Proposition 3.7.6, x is a log-atomic number. Consider the surreal number

$$y = \left[\left\{ y' \in \mathbf{No} \,\middle|\, \begin{array}{cc} & \lambda_{y'} \prec^L x \\ \exists z \sqsubset x & \lambda_{y'} \asymp^L z \end{array} \right\} \,\middle|\, \left\{ y'' \in \mathbf{No} \,\middle|\, \begin{array}{cc} & x \prec^L \lambda_{y''} \\ \exists z \sqsubset x & \lambda_{y'} \asymp^L z \end{array} \right\}\right]$$

By assumption on x, $\lambda_y \not\asymp^L x$. But since $\lambda_{y'} \prec^L x \prec^L \lambda_{y''}$ by simplicity we get that $\lambda_y \sqsubset x$. But this contradicts the definition of y, therefore $\lambda_y \asymp^L x$. We conclude using Proposition 3.7.6. ☐

Corollary 3.7.17 ([12, Berarducci and Mantova, Corollary 5.17]). *The λ-map is a parametrization of all the log-atomic numbers:* $\mathbb{L} = \lambda_{\mathbf{No}}$.

Remark 3.7.18. $\mathbb{L}$ is a surreal substructure such that $\Xi_{\mathbb{L}} = \lambda_{(\cdot)}$.

An interesting thing we can notice is that exponential and logarithm behave very nicely with these numbers.

Proposition 3.7.19 ([6, Aschenbrenner, van den Dries and van der Hoeven, Proposition 2.5]). *For all surreal number x,*

$$\exp \lambda_x = \lambda_{x+1} \qquad \textit{and} \qquad \ln \lambda_x = \lambda_{x-1}$$

The proof of the previous proposition is just a simple proof by induction using the uniformity property. With the same idea we can give some interesting examples.

Lemma 3.7.20 ([6, Aschenbrenner, van den Dries and van der Hoeven, Lemma 2.6]). *For all ordinal α, $\lambda_{-\alpha} = \omega^{\omega^{-\alpha}}$.*

3.7.4 Relation between the λ-map and the κ-map

As noticed in [6], from [33] we can derive

Lemma 3.7.21 ([6, Aschenbrenner, van den Dries and van der Hoeven, Corollary 2.9]). *For all ordinal number α,*

$$\kappa_{-\alpha} = \lambda_{-\omega\otimes\alpha} = \omega^{\omega^{-\omega\otimes\alpha}}$$

Lemma 3.7.22 ([9, Bagayoko and van der Hoeven]). *For all $a \in \mathbf{No}$, $\kappa_a = \lambda_{\omega \dot{\times} a}$.*

Proof. From [9, Proposition 7.2], any purely infinite number x has form $\omega\dot{\times}a$. They also follow Matova and Matusinsky to show that λ_x is a κ-number if an only if x is purely infinite. The equality follows from the fact the isomorphism between surreal substructure is unique. ☐

Remark 3.7.23. The operation $\dot{\times}$ is defined in [9]. It is, roughly speaking, an equivalent of the multiplication in the context where the addition is a concatenation.

3.8 Tree representation

Surreal numbers have a natural representation as well-founded trees. As always when dealing with such a structure, there is an underlying well partial order. In the framework of surreal numbers, this is called the nested truncation rank and is a generalization of the rank over transseries (see section 4.2.2).

3.8.1 Nested truncation rank

The nested truncation rank has been defined by Berarducci and Mantova [12]. It is associated to a well partial order $\trianglelefteq$.

Definition 3.8.1 ([12, Berarducci and Mantova, Definition 4.3]). For all natural number $n \in \mathbb{N}$, we define the relation $\trianglelefteq_n$ as follows:

- Writing $y \trianglelefteq_0 x$ if any only if $y = \sum_{i<\nu'} r_i\omega^{a_i}$ and $x = \sum_{i<\nu} r_i\omega^{a_i}$ with $\nu' \leq \nu$. We say that y is a **truncation** of x.

- Let $x = \sum_{i<\nu} r_i\omega^{a_i}$ Since $\omega^{\mathbf{No}} = \exp(\mathbf{No}_\infty)$, we can write

  $$x = \sum_{i<\nu} r_i \exp(x_i)$$

  where $\exp(x_i) = \omega^{a_i}$. For a surreal number y, we say $y \trianglelefteq_{n+1} x$ if there is $\nu' < \nu$ and $y' \trianglelefteq_n x_{\nu'}$ such that

  $$y = \sum_{i<\nu'} r_i \exp(x_i) + \operatorname{sign}(r_{\nu'}) \exp y'$$

  We say that y is a **nested truncation** of x.

We also write $y \trianglelefteq x$ is there is some natural number n such that $y \trianglelefteq_n x$. We also introduce the corresponding strict relations $\vartriangleleft_n$ and $\vartriangleleft$.

Proposition 3.8.2 ([12, Berarducci and Mantova, Proposition 4.8 and Theorem 4.26]). *The relation $\trianglelefteq$ is a well-partial order over $\mathbf{No}^*$ and $\vartriangleleft$ is the corresponding strict order.*

Note that this proposition, even if it is intuitive is not trivial. To prove it is necessary to handle monomial carefully and simplicity relations. This proposition enables us to define a corresponding rank:

Definition 3.8.3 (Nested truncation rank [12, Berarducci and Mantova, Definition 4.27]). The **nested truncation rank** of $x \in \mathbf{No}^*$ is defined by

$$\mathrm{NR}(x) = \sup \{ \mathrm{NR}(y) + 1 \mid y \triangleleft x \}$$

By convention, we also set $\mathrm{NR}(0) = 0$.

3.8.2 Surreal numbers as trees

Each surreal number has a natural structure of tree. This representation behaves very nicely with the notion of nested truncation rank described in the previous section. We first recall what we means by "tree".

Definition 3.8.4 (Tree). A **well-founded tree** or simply **tree** is a (partially) ordered set $(T, \leq)$ such that for any $s \in S$, the set $\{ t \in T \mid t < s\}$ is totally ordered and finite and such that there is some $r \in T$ (called the **root**) such that $r \leq t$ for all $t \in T$.

Remark 3.8.5. Note that if $s < t$ are elements of a tree T, there are finitely many u (if any) such that $s < u < t$. Indeed, if it was not the case, t would fail the definition of the tree T.

Thanks to the previous remark, we state:

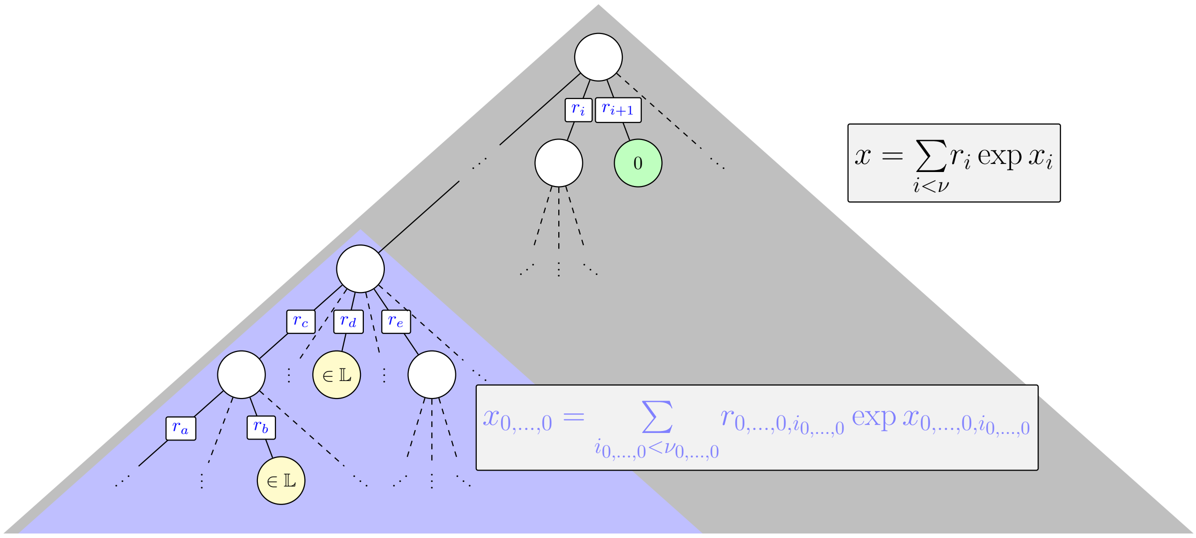
Definition 3.8.6. Let T be a tree and $t \in T$. If $s \in T$ is such that $s < t$, s is called an **ancestor** of t and t is a **descendant** of s. If s is the largest (which exists because of the remark above) ancestor of t, then s is the **parent** of t and t is the **child** of s.

Definition 3.8.7 (Well-ordered tree). An **well-ordered tree** $(T, \leq_1, \leq_2)$ is a structure such that $(T, \leq_1)$ is a tree and $\leq_2$ is a total order over T such that for any $t \in T$, the children of t are well-ordered by $\leq_2$.

Definition 3.8.8 (Well-ordered tree representation). A **well-ordered tree representation** of a well-ordered tree $(T, \leq_1, \leq_2)$ is an (oriented) graph whose vertices are labeled by elements of T and such that (u, v) is an edge is v is a child of u in T. We represent such a tree by putting the root at the top and order children from left to right according to $\leq_2$ below their parent (spatial rule). It can come with a label function ℓ that labels the edges.

Since there is not ambiguity due to the spatial rule of the definition, we may represent the graph as a non-oriented graph. However, if we do not necessarily apply the spatial rule, we need to make them explicit.
We are now ready to show that surreal number have a natural well-ordered tree representation.

Definition 3.8.9 (Well-ordered tree representation of surreal number). Let $x \in \mathbf{No}$ be a surreal number. Then x has a natural well-ordered tree representation given by the following graph :

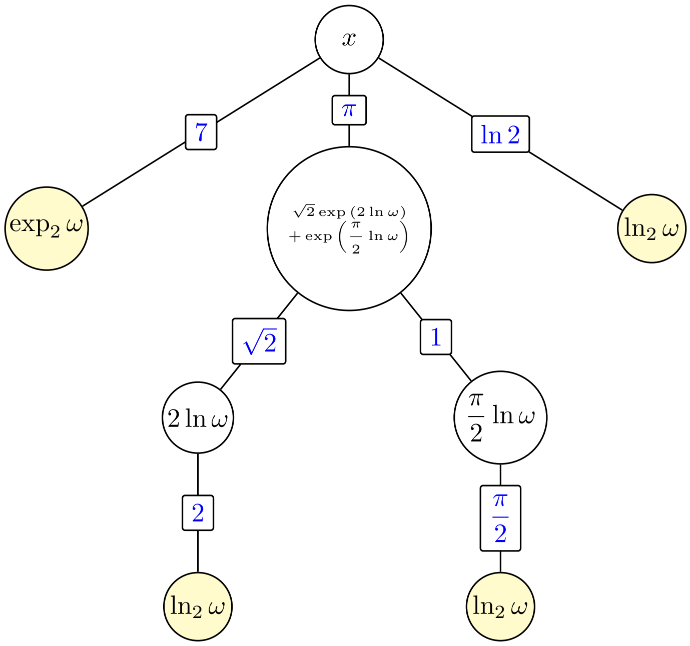


The surreal number x is at the root. If u is non-log-atomic and is a node and if v is purely infinite such that $r \exp v$ is a term of u for some $r_i n \mathbb{R}^*$, then (u, v) is an edge labeled by r. If u is log-atomic, it must be a leaf. If u is 0, it must be either the root and the only node, either a child of the root.

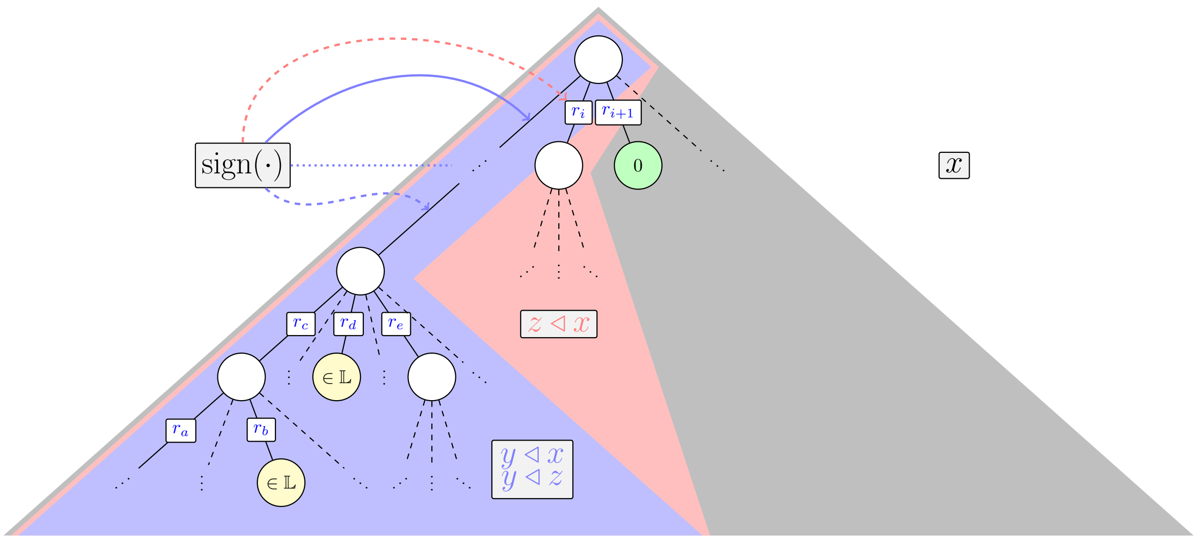
Example 3.8.10. For the surreal number

$$x = 7\omega^{\omega^\omega} + \pi\omega^{\sqrt{2}\omega^2 + \omega^{\frac{\pi}{2}}} + \ln 2\omega^{\frac{1}{\omega}} = 7\exp_2 \omega + \pi \exp\left(\sqrt{2}\exp(2\ln\omega) + \exp\left(\frac{\pi}{2}\ln\omega\right)\right) + (\ln 2)\ln\omega$$

the tree representation is:



It is possible to read the nested truncation rank from the tree representation. Indeed, up to changing some of the labels to their respective sign, every y such that $y \triangleleft x$ has its tree representation included into the one of x. This can be seen on the figure below:



The dotted arrows from "sign" are to be understood by the fact that we can apply the sign function or not to this arrow. The plain one means that we must apply it. Thanks to this figure we can understand $y \triangleleft x$ by the fact that the tree representation of y is a left-part of the tree representation of x.

Remark 3.8.11. The reason why we stop the construction on log-atomic numbers is because if we proceed the construction, we would get an infinite path where each node as exactly one child and where every edge is labeled by 1.

This notion of tree comes with a notion of path inside the tree.

Definition 3.8.12. Let x be a surreal number. A **path** P of x is sequence $P : \mathbb{N} \to \mathbf{No}$ such that

- $P(0)$ is a term of x
- For all $i \in \mathbb{N}$, $P(i+1)$ is an infinite term of $\ln |P(i)|$

We denote $\mathcal{P}(x)$ the set of all paths of x.

Definition 3.8.13. The **dominant path** of x is the path such that

- $P(0)$ is the leading term of x
- $P(i+1)$ is the leading term of $\ln |P(i)|$.

In a more graphical point of view, the dominant path of x is the left most path in the tree of x that does not end on the leaf 0. This reduce to the left most path if $x \not\asymp 1$.

Notation. For P a path and integer $k \in \mathbb{N}$, we denote $P[k:]$ (Python–like notation) the path defined by

$$\forall m \in \mathbb{N} \qquad P[k:](m) = P(m+k)$$

3.8.3 Properties of the nested truncation rank

This section investigates some properties of the nested truncation rank. More precisely, we provide compatibility properties with the operations over surreal numbers and bounds on some particular nested truncation ranks.

Proposition 3.8.14 ([12, Berarducci and Mantova, Proposition 4.28])**.** *If $\gamma \in \mathbf{No}_\infty$, then* $\mathrm{NR}(\pm\exp\gamma) = \mathrm{NR}(\gamma)$.

Corollary 3.8.15. *For all $a \in \mathbf{No}^*$,* $\mathrm{NR}(a) = \mathrm{NR}\,(-a)$

Proof. Without loss of generality, we assume that $a > 0$. Then

$$\begin{aligned}\mathrm{NR}\,(a) &= \mathrm{NR}\,(\ln a) && \text{(Proposition 3.8.14)}\\ &= \mathrm{NR}(-\exp\ln a) && \text{(Proposition 3.8.14)}\\ &= \mathrm{NR}\,(-a)\end{aligned}$$

□

Corollary 3.8.16. *For all $a \in \mathbf{No}^*$,* $\mathrm{NR}(a) = \mathrm{NR}\left(\frac{1}{a}\right)$

Proof.

$$\begin{aligned}\mathrm{NR}\left(\frac{1}{a}\right) &= \mathrm{NR}\left(\ln\frac{1}{a}\right) && \text{(Proposition 3.8.14)}\\ &= \mathrm{NR}(-\ln a)\\ &= \mathrm{NR}\,(\ln a) && \text{(Corollary 3.8.15)}\\ &= \mathrm{NR}(a) && \text{(Proposition 3.8.14)}\end{aligned}$$

□

Lemma 3.8.17. *For all $x \in \mathbf{No}$,* $\mathrm{NR}(x) = 0$ *iff either $x \in \mathbb{R}$ or $x = \pm\lambda^{\pm 1}$ for some log-atomic number λ.*

Proof. (SC ⇐) Note that if $x \in \mathbb{R}$ then there is no $y \in \mathbf{No}$ such that $y \lhd x$. Therefore $\mathrm{NR}(x) = 0$. Now assume that there is some $x = \pm\lambda^{\pm 1}$ with $\lambda \in \mathbb{L}$ such that $\mathrm{NR}(x) \neq 0$. Therefore there is some $y \in \mathbf{No}$ such that $y \lhd x$. Let $n \in \mathbb{N}$ minimal such that there is $y \in \mathbf{No}$ and $\lambda \in \mathbb{L}$ such that $y \lhd_n \pm\lambda^{\pm 1}$. Note that since $\pm\lambda^{\pm 1}$ is a term, $n > 0$. Then $y = \pm\exp(\pm y')$ with $y' \lhd_{n-1} \ln\lambda \in \mathbb{L}$. But this contradicts the minimality of n. hence, for all $\lambda \in \mathbb{L}$, $\mathrm{NR}\left(\pm\lambda^{\pm 1}\right) = 0$.

(NC ⇒) Assume $\mathrm{NR}(x) = 0$ and x is not a real number. If x is not a term, then there is $y \lhd_0 x$ and in particular $\mathrm{NR}(x) \geq 1$, what is impossible. Therefore there is some $r \in \mathbb{R}^*$ and some $x_1 \in \mathbb{J}$ such that $x = r\exp(x_1)$. If $r \neq \pm 1$ then $\mathrm{sign}(x)\exp(x') \lhd x$ what is again impossible. Hence, $x = \pm\exp(x_1)$. Proposition 3.8.14 ensures that $\mathrm{NR}(x_1) = 0$. We then can apply the same work to x_1 so that there is some $x_2 \in \mathbb{J}$ such that $x_1 = \pm\exp(x_2)$. By induction, we can always define $x_n = \pm\exp(x_{n+1})$ with $x_{n+1} \in \mathbb{J}$. For $n \geq 1$ we have $x_n \in \mathbb{J}$, therefore $x_{n+1} > 0$. In particular

$$\forall n \geq 2 \qquad x_n = \exp(x_{n+1})$$

So, for all $n \in \mathbb{N}$, $\ln_n x_2$ is a monomial, this means that $x_2 \in \mathbb{L}$. We also have

$$x = \pm\exp\left(\pm\exp x_2\right) = \pm\left(\exp_2(x_2)\right)^{\pm 1}$$

Since $\exp_2 x_2 \in \mathbb{L}$, we have the expected result.

□

Lemma 3.8.18. *Let $x = \sum_{i<\nu} r_i\omega^{a_i}$ and $r \in \mathbb{R}^*$, $a \in \mathbf{No}$ such that for all $i < \nu$, $r\omega^a \prec \omega^{a_i}$. Then*

$$\mathrm{NR}(x + r\omega^a) = \mathrm{NR}(x) \oplus 1 \oplus \mathrm{NR}(\omega^a) \oplus \mathbb{1}_{r\neq\pm 1}$$

where the $\oplus$ is the usual sum over ordinal numbers.

Proof. Let $y \lhd x + r\omega^a$. Then $y \unlhd x$ or $y = x + \operatorname{sign}(r)\exp(\delta)$ with $\delta \lhd \ln \omega^a$ or, if $r \neq \pm 1$, $y = x + \operatorname{sign}(r)\omega^a$. Let

$$A = \{ y \mid y \unlhd x \} \qquad \text{and} \qquad B = \{ x + \operatorname{sign}(r)\exp(\delta) \mid \delta \lhd \ln \omega^a \}$$

and
$$C = \begin{cases} \varnothing & r = \pm 1 \\ x + \operatorname{sign}(r)\omega^a & r \neq \pm 1 \end{cases}$$

One can easily see that $\quad \forall y \in A \quad \forall y' \in B \quad \forall y'' \in C \qquad y \lhd y' \wedge y \lhd y'' \wedge y' \lhd y''$

We now proceed by induction on $\mathrm{NR}(\omega^a)$.

- If $\mathrm{NR}(\omega^a) = 0$, using Lemma 3.8.17, either $\omega^a = \pm\lambda^{\pm 1}$ for some log-atomic number λ or $a = 0$. In both cases, there is no $\delta \lhd \ln \omega^a$.

$$\begin{aligned}
\mathrm{NR}(x + r\omega^a) &= \sup \{ \mathrm{NR}(y) + 1 \mid y \in A \cup C \} \\
&= \sup \left(\left\{ \underbrace{\mathrm{NR}(y) + 1}_{\leq \mathrm{NR}(x)} \,\middle|\, y \lhd x \right\} \cup \{ \mathrm{NR}(x) + 1 \} \cup \{ \mathrm{NR}(y) + 1 \mid y \in C \} \right) \\
&= \begin{cases} \mathrm{NR}(x) + 1 & r = \pm 1 \\ \mathrm{NR}(x + \operatorname{sign}(r)\omega^a) & r \neq \pm 1 \end{cases} \\
&= \begin{cases} \mathrm{NR}(x) + 1 & r = \pm 1 \\ \mathrm{NR}(x) + 2 & r \neq \pm 1 \end{cases} \\
\mathrm{NR}(x + r\omega^a) &= \mathrm{NR}(x) + 1 + \mathrm{NR}(\omega^a) + \mathbb{1}_{r \neq \pm 1}
\end{aligned}$$

- For heredity now. Let $\delta \lhd \ln \omega^a$. Since $\ln \omega^a$ is a purely infinite number, so is δ. Then $\exp \delta$ is of the form ω^b for some surreal $b \in \mathbf{No}$. Moreover

$$\mathrm{NR}(\omega^b) \underset{\text{Proposition 3.8.14}}{=} \mathrm{NR}(\delta) < \mathrm{NR}(\ln \omega^a) \underset{\text{Proposition 3.8.14}}{=} \mathrm{NR}(\omega^a)$$

  From the induction hypothesis, we have that for any $\delta \lhd \ln \omega^a$

$$\mathrm{NR}(x + \operatorname{sign}(r)\exp(\delta)) = \mathrm{NR}(x) \oplus 1 \oplus \mathrm{NR}(\exp \delta)$$

  Now
$$\begin{aligned}
\mathrm{NR}(x + r\omega^a) &= \sup \{ \mathrm{NR}(y) + 1 \mid y \in B \cup C \} \\
&= \sup \left(\left\{ \underbrace{\mathrm{NR}(x + \operatorname{sign}(r)\exp\delta) + 1}_{\leq \mathrm{NR}(x + \operatorname{sign}(r)\omega^a)} \,\middle|\, \delta \lhd \ln \omega^a \right\} \cup \{ \mathrm{NR}(y) + 1 \mid y \in C \} \right) \\
&= \sup \{ \mathrm{NR}(x) \oplus 1 \oplus \mathrm{NR}(\exp \delta) \oplus 1 \mid \delta \lhd \ln \omega^a \} + \mathbb{1}_{r \neq \pm 1} \\
&= \mathrm{NR}(x) \oplus 1 \oplus \sup \{ \mathrm{NR}(\exp \delta) + 1 \mid \delta \lhd \ln \omega^a \} \oplus \mathbb{1}_{r \neq \pm 1} \\
\mathrm{NR}(x + r\omega^a) &= \mathrm{NR}(x) \oplus 1 \oplus \mathrm{NR}(\omega^a) \oplus \mathbb{1}_{r \neq \pm 1}
\end{aligned}$$

□

Lemma 3.8.19. *Let $x = \sum_{i<\nu} r_i \omega^{a_i}$ such that for all $i < \nu$, $r_i = \pm 1$ and $\omega^{a_i} = \lambda_i^{\pm 1}$ for some $\lambda \in \mathbb{L}$. Then*

$$\mathrm{NR}(x) = \begin{cases} \nu + 1 & \nu < \omega \\ \nu & \nu \geq \omega \end{cases}$$

Proof. If $\nu < \omega$, we just proceed by induction using Lemma 3.8.18. Now we prove by induction the remaining.

- If $\nu = \omega$. Then

$$\mathrm{NR}(x) = \sup \left\{ \mathrm{NR}\left(\sum_{i<\nu'} r_i \omega^{a_i} \right) + 1 \,\middle|\, \nu' < \nu \right\} = \sup \{ \nu' + 2 \mid \nu' < \omega \} = \omega$$

- Assume for $\omega \leq \nu' < \nu$, $\mathrm{NR}\left(\sum_{i<\nu'} r_i \omega^{a_i} \right) = \nu'$. If ν is a non-limit ordinal, then Lemma 3.8.18 concludes. Otherwise

$$\mathrm{NR}(x) = \sup \left\{ \mathrm{NR}\left(\sum_{i<\nu'} r_i \omega^{a_i} \right) + 1 \,\middle|\, \nu' < \nu \right\} = \sup \left\{ \nu' +^1 \,\middle|\, \omega \leq \nu' < \nu \right\} = \nu$$

□

Lemma 3.8.20. *Let $x = \sum_{i<\nu} r_i \omega^{a_i} \in \mathbf{No}$. Then $\nu \leq \mathrm{NR}(x) + 1$. The equality stands iff x is a finite sum of numbers of the form $\pm y^{\pm 1}$ with $y \in \mathbb{L}$ and possibly one non-zero real number.*

Proof. Using induction on ν it is trivial. For 0, $\nu = 0 = \mathrm{NR}(0)$. Now assume $\nu \neq 0$. Then, by definition

$$\mathrm{NR}(x) + 1 \geq \sup\{\mathrm{NR}(y) + 1 \mid y \lhd_0 x \quad y \neq 0\} + 1 \underset{\text{induction hypothesis}}{\geq} \sup\{\nu(y) \mid y \lhd_0 x \quad y \neq 0\} + 1 \geq \nu(x)$$

Now assume $\nu(x) = \mathrm{NR}(x) + 1$ and write $x = \sum_{i<\nu(x)} r_i\omega^{a_i}$. We use induction on $\mathbf{No}^*$ with the well partial order $\lhd_0$.

- If x is a monomial, $\nu(x) = 1$ and $\mathrm{NR}(x) = 0$. That is $x = \pm y^{\pm 1}$ for some $y \in \mathbb{L}$ or $x \in \mathbb{R}$ (using Lemma 3.8.17).
- If x is not a monomial. Assume $r_i\omega^{a_i} \notin \pm\mathbb{L}^{\pm 1} \cup \mathbb{R}^*$ with i minimal for that property. Let $x' = \sum_{j<i} r_j\omega^{a_j}$.
  - ➢ If $i = 0$ then $\mathrm{NR}(r_0\omega^{a_0}) \geq 1$. A simple induction shows that $\mathrm{NR}\left(\sum_{i<\nu'} r_i\omega^{a_i}\right) \geq \nu'$ for all $\nu' \leq \nu$. What is a contradiction.
  - ➢ Otherwise $x' \neq 0$ and $x' \lhd_0 x$. If $\mathrm{NR}(x') + 1 \neq i$ then $\mathrm{NR}(x') \geq i$ and
  $$\mathrm{NR}(x) \geq \mathrm{NR}(x') \oplus (\nu \ominus i) \geq \nu$$
  where $\nu \ominus i$ is the ordinal such that $i \oplus (\nu \ominus i) = \nu$. what is a contraction. Then by induction hypothesis, $i = \mathrm{NR}(x') + 1$ is finite. Now consider $y \lhd x' + r_i\omega^{a_i}$. Then $y \trianglelefteq_0 x'$ ($y \lhd_n x'$ with $n \geq 1$ is impossible since x' has only terms in $\pm\mathbb{L}^{\pm 1} \cup \mathbb{R}$) or $y = x' + \mathrm{sign}(r_i)\exp(\delta)$ with $\delta \trianglelefteq \ln(\omega^{a_i})$. Since $r_i\omega^{a_i} \notin \pm\mathbb{L}^{\pm 1} \cup \mathbb{R}$, there is such a y of the later form such that $y \neq x' + r_i\omega^{a_i}$. From Lemma 3.8.18, we have $\mathrm{NR}(y) \geq \mathrm{NR}(x') + 1$. Then $\mathrm{NR}(x' + r_i\omega^{a_i}) \geq \mathrm{NR}(y) + 1 \geq \mathrm{NR}(x') + 2$. By induction we then can show that
  $$\mathrm{NR}(x) \geq \mathrm{NR}(x' + r_i\omega^{a_i}) \oplus (\nu - (i+1)) \geq \mathrm{NR}(x') \oplus 2 \oplus (\nu \ominus (i \oplus 1)) = i \oplus 1 + (\nu \ominus (i \oplus 1)) = \nu$$
  and we get a contradiction.

  Then, every term of x is in $\pm\mathbb{L}^{\pm 1} \cup \mathbb{R}$ and by definition only one can be a non-zero real number. It remains to show that there are finitely many terms, what follows from Lemma 3.8.19.

□

Remark 3.8.21. For all $x \in \mathbf{No}$, $\mathrm{NR}(x) \leq |x|_{+-}$

Proof. Assume the converse and take x with minimal length that contradicts the property then there is $y \lhd x$ such that $\mathrm{NR}(y) \geq |x|_{+-}$. Since $|x|_{+-} > |y|_{+-}$, then y reaches contradiction with the minimality of x. □

Proposition 3.8.22 ([12, Berarducci and Mantova, Proposition 4.29]). *For all $a \in \mathbf{No}^*$, for all $r \in \mathbb{R} \setminus \{\pm 1\}$, we have* $\mathrm{NR}(r\omega^a) = \mathrm{NR}(\omega^a) + 1$.

Proposition 3.8.23 ([12, Berarducci and Mantova, Proposition 4.30]). *Let $x = \sum_{i<\nu} r_i\omega^{a_i} \in \mathbf{No}^*$. Then*

- $\forall i < \nu \qquad \mathrm{NR}(r_i\omega^{a_i}) \leq \mathrm{NR}(x)$
- $\forall i < \nu \qquad i + 1 < \nu \Rightarrow \mathrm{NR}(r_i\omega^{a_i}) < \mathrm{NR}(x)$

We can also say something about the nested truncation rank of a sum of surreal number.

Lemma 3.8.24. *For $a, b \in \mathbf{No}$, $\mathrm{NR}(a+b) \leq \mathrm{NR}(a) + \mathrm{NR}(b) + 1$ (natural sum of ordinal, which correspond to the surreal sum).*

Proof. We prove it by induction on the ordered pair $(\mathrm{NR}(a), \mathrm{NR}(b))$.

- If $\mathrm{NR}(a) = \mathrm{NR}(b) = 0$ then, by Lemma 3.8.17 both a, b are in $\pm\mathbb{L}^{\pm 1} \cup \mathbb{R}$. If $a \in \mathbb{R}$ or $b \in \mathbb{R}$ then $\mathrm{NR}(a+b) \leq 1$ by Lemmas 3.8.18 and 3.8.17. Otherwise, either $a = \pm b$ and then $\mathrm{NR}(a+b) = 0$ or $a \neq \pm b$ and Lemma 3.8.20 ensure that $\mathrm{NR}(a+b) = 1$.
- Assume the property for all x, y such that $(\mathrm{NR}(x), \mathrm{NR}(y)) <_{lex} (\mathrm{NR}(a), \mathrm{NR}(b))$. Then consider $y \lhd a + b$. Write $a + b = \sum_{i<\nu} r_i\omega^{a_i}$.
  - ➢ If $y = \sum_{i<\nu'} r_i\omega^{a_i}$ with $\nu' < \nu$. Let z_a be the series constituted of the terms of a which asolute value is infintely larger than $\omega^{a'_\nu}$. We define the same way z_b. Then $y = z_a + z_b$. We have $(\mathrm{NR}(z_a), \mathrm{NR}(z_b)) <_{lex} (\mathrm{NR}(a), \mathrm{NR}(b))$ since there is term with order of magnitude $\omega^{a_{\nu'}}$ in either a or b. Then, applying the induction hypothesis,
  $$\mathrm{NR}(y) \leq \mathrm{NR}(z_a) + \mathrm{NR}(z_b) + 1$$
  Since we have at least one of the following inequalities $z_a \lhd_0 a$ or $z_b \lhd_0 b$, then $\mathrm{NR}(z_a) + 1 \leq \mathrm{NR}(a)$ or $\mathrm{NR}(z_b) + 1 \leq \mathrm{NR}(b)$. In all cases
  $$\mathrm{NR}(y) + 1 \leq \mathrm{NR}(a) + \mathrm{NR}(b) + 1$$

➢ If $y = \sum_{i<\nu'} r_i\omega^{a_i} + \operatorname{sign}(r_{\nu'})\exp(y')$ with $\nu' < \nu$ and $y' \trianglelefteq \ln\omega^{a_{\nu'}}$ (and $y \triangleleft \ln\omega^{a_{\nu'}}$ if $r_{\nu'} = \pm 1$). Let z_a be the series constitued of the terms of a which absolute value is infinitely larger than $\omega^{a'_\nu}$. We define the same way z_b. Then $y = z_a + z_b + \operatorname{sign}(r_{\nu'})\omega^{a_{\nu'}}$. Since there is term with order of magnitude $\omega^{a_{\nu'}}$ with the same sign as $r_{\nu'}$ in either a or b. Without loss of generality, assume it is a. Then $z_a + \operatorname{sign}(r_{\nu'})\exp y' \trianglelefteq a$. We have $(\mathrm{NR}(z_a + \operatorname{sign}(r_{\nu'})\exp y'), \mathrm{NR}(z_b)) <_{lex} (\mathrm{NR}(a), \mathrm{NR}(b))$ otherwise $y = a + b$ what is not the case. Then, applying the induction hypothesis,

$$\mathrm{NR}(y) \leq \mathrm{NR}(z_a + \operatorname{sign}(r_{\nu'})\exp y') + \mathrm{NR}(z_b) + 1$$

Since we have at least one of the following inequalities $z_a + \operatorname{sign}(r_{\nu'})\exp y' \triangleleft a$ or $z_b \triangleleft_0 b$, then we have either $\mathrm{NR}(z_a + \operatorname{sign}(r_{\nu'})\exp y') + 1 \leq \mathrm{NR}(a)$ or $\mathrm{NR}(z_b) + 1 \leq \mathrm{NR}(b)$. In all cases

$$\mathrm{NR}(y) + 1 \leq \mathrm{NR}(a) + \mathrm{NR}(b) + 1$$

Then, for any $y \triangleleft a + b$, $\mathrm{NR}(y) + 1 \leq \mathrm{NR}(a) + \mathrm{NR}(b) + 1$. This proves that

$$\mathrm{NR}(a + b) \leq \mathrm{NR}(a) + \mathrm{NR}(b) + 1$$

□

Corollary 3.8.25. *For all* $a, b \in \mathbf{No}$, $\mathrm{NR}(ab) \leq \mathrm{NR}(a) + \mathrm{NR}(b) + 1$.

Proof.

We have
$$\begin{aligned} \mathrm{NR}(ab) &= \mathrm{NR}(\ln(ab)) && \text{(Proposition 3.8.14)}\\ &= \mathrm{NR}(\ln a + \ln b) \\ &\leq \mathrm{NR}(\ln a) + \mathrm{NR}(\ln b) + 1 && \text{(Lemma 3.8.24)}\\ &\leq \mathrm{NR}(a) + \mathrm{NR}(b) + 1 && \text{(Proposition 3.8.14)} \end{aligned}$$

□

3.9 Derivation on surreal numbers

3.9.1 Properties of a general derivation

Following Berarducci and Mantova [12], we define a derivation over the class-field **No**. The reader may be aware that it is a derivation of the numbers themselves and not of function over them. We make the definition slightly more general since we define it over an arbitrary field ordered $\mathbb{K}$ which may be a class-field. We will apply it for subfield of **No** and the classical exponential function over **No** restricted to $\mathbb{K}$.

Definition 3.9.1 (Summable family)**.** Let $\{x_i\}_{i\in I}$ be a family of surreal numbers. For $i \in I$ write

$$x_i = \sum_{a\in\mathbf{No}} r_{i,a}\omega^a$$

The family $\{x_i\}_{i\in I}$ is **summable** *iff*

(i) $\bigcup_{i\in I} \operatorname{supp} x_i$ is a reverse well ordered set.

(ii) For all $a \in \bigcup_{i\in I} \operatorname{supp} x_i$, $\{\, i \in I \mid a \in \operatorname{supp} x_i\}$ is a finite set.

In this case, its sum is defined as $\sum_{i\in I} x_i = \sum_{a\in\mathbf{No}} s_a\omega^a$ where for all $a \in \mathbf{No}$,

$$s_a = \sum_{i\in I \,\mid\, a\in\operatorname{supp} x_i} r_{i,a}$$

which is a finite sum.

Definition 3.9.2 ([12, Berarducci and Mantova, Definition 6.1])**.** A **derivation** D over a totally ordered exponential (class)-field $\mathbb{K} \supseteq \mathbb{R}$ is a function $D : \mathbb{K} \to \mathbb{K}$ such that

D1. It satisfies $$\forall x, y \in \mathbb{K} \qquad D(xy) = xD(y) + D(x)y \qquad \text{(Liebniz Rule)}$$

D2. If $\{x_i\}_{i\in I}$ is summable, $$D\left(\sum_{i\in I} x_i\right) = \sum_{i\in I} D(x_i) \qquad \text{(Strong additivity)}$$

D3. $\forall x \in \mathbb{K} \qquad D(\exp x) = D(x)\exp x$

D4. $\ker D = \mathbb{R}$

D5. $\forall x > \mathbb{N} \qquad D(x) > 0$

Remark 3.9.3. We can replace Axiom **D2.** by

D2'. If $\{x_i\}_{i\in I}$ is summable and $\{r_i\}_{i\in I}$ is a family of real numbers,

$$D\left(\sum_{i\in I} r_i x_i\right) = \sum_{i\in I} r_i D(x_i) \qquad \text{(Strong linearity)}$$

Indeed we have **D2'.** $\implies$ **D2.** and **D1.** $\wedge$ **D2.** $\wedge$ **D4.** $\implies$ **D2'.**

Proposition 3.9.4 ([12, Berarducci and Mantova, Proposition 6.4])**.** *We have the following properties :*

- $\forall x, y \in \mathbb{K} \qquad 1 \not\asymp x \succ y \Rightarrow D(x) \succ D(y)$
- $\forall x, y \in \mathbb{K} \qquad 1 \not\asymp x \sim y \Rightarrow D(x) \sim D(y)$
- $\forall x, y \in \mathbb{K} \qquad 1 \not\asymp x \asymp y \Rightarrow D(x) \asymp D(y)$

If $\mathbb{K} \subseteq \mathbf{No}$ is stable under $\exp$ and $\ln$, we can get a nice property satisfied by a general derivation.

Proposition 3.9.5 ([12, Berarducci and Mantova, Proposition 6.5])**.** *Let $\mathbb{K} \subseteq \mathbf{No}$ be a field of surreal number stable by* $\exp$ *and* $\ln$*. Let D be a derivation over $\mathbb{K}$. For all $x, y > \mathbb{N}$ such that $x - y > \mathbb{N}$,*

$$\ln D(x) - \ln D(y) \prec x - y \preceq \max(x, y)$$

Remark 3.9.6. The second inequality in the previous proposition is trivial.

The key point to define a surreal derivation, is to identify the basic numbers on with it is easy to define a derivation. Since we want a compatibility with the exponential function, these basic number will be numbers that does not really change under $\ln$ and $\exp$, log-atomic numbers. The previous proposition is crucial to propagate the definition of a derivation. Therefore we will request it to be satisfied for log-atomic numbers.

3.9.2 General derivation from a derivation over log-atomic numbers

To define a derivation, we will use a definition over a base case, the log-atomic numbers, and propagate the definition of the derivative with paths. In this section we assume that we are given a derivation defined only on log-atomic numbers, denoted $\partial_{\mathbb{L}}$, which is called a prederivation.

Definition 3.9.7 ([12, Berarducci and Mantova, Definition 9.1] Prederivation)**.** Let $\mathbb{K}$ be a field of surreal numbers stable under $\exp$ and $\ln$ and such that for all $x \in \mathbb{K}$, for all path $P \in \mathcal{P}(x)$, for all $k \in \mathbb{N}$, if $P(k) \in \mathbb{L}$, then $P(k) \in \mathbb{K}$. A **prederivation** over $\mathbb{K}$ is a function $D_{\mathbb{L}} : \mathbb{L} \cap \mathbb{K} \to \mathbb{K}$ such that

D3. $\forall \lambda \in \mathbb{L} \cap \mathbb{K} \qquad D_{\mathbb{L}} \exp \lambda = (D_{\mathbb{L}}\lambda) \exp \lambda$

PD1. For all $\lambda \in \mathbb{L} \cap \mathbb{K}$, $D_{\mathbb{L}}\lambda$ is a positive term.

PD2. $\forall \lambda, \mu \in \mathbb{L} \cap \mathbb{K} \qquad \ln D_{\mathbb{L}}\lambda - \ln D_{\mathbb{L}} \ln \mu \prec \max(\lambda, \mu)$

We can extend prederivations to a derivation over the whole field. To do so, we introduce the path derivative. For now on we assume we are given a prederivation $D_{\mathbb{L}}$ over a field $\mathbb{K}$ of surreal numbers. The path derivative is defined as follows :

Definition 3.9.8 ([12, Berarducci and Mantova, Definition 6.13] Path derivative)**.** Let P be a path. We define the **path derivative** $\partial P \in \mathbb{R}\omega^{\mathbf{No}}$ by

$$\partial P = \begin{cases} P(0)\cdots P(k-1) D_{\mathbb{L}} P(k) & P(k) \in \mathbb{L} \\ 0 & \forall k \in \mathbb{N} \quad P(k) \notin \mathbb{L} \end{cases}$$

Remark 3.9.9. The value of ∂P does not depend of the choice of k in the first case since, because of Axiom **D3.**, the derivative over log-atomic numbers satisfies $D_{\mathbb{L}}P(k) = P(k) D_{\mathbb{L}} P(k+1)$ whenever $P(k) \in \mathbb{L}$.

Notation. We denote $\mathcal{P}_{\mathbb{L}}(x) = \{\, P \in \mathcal{P}(x) \mid \partial P \neq 0 \}$
For $P \in \mathcal{P}_{\mathbb{L}}(x)$, we define k_P the smallest integer such that $P(k_P) \in \mathbb{L}$.

Notation. For a non-zero surreal number x, we denote $\ell(x)$ to be the purely infinite part of $\ln |x|$.

One can notice that for any $P \in \mathcal{P}_{\mathbb{L}}(x)$, $\partial P = r\omega^a$ for some $r \in \mathbb{R}^*$ and $a \in \mathbf{No}$. Indeed, every $P(k)$ is a term and $D_{\mathbb{L}}P(k)$, when $P(k) \in \mathbb{L}$, is an exponential of a purely infinite number, hence, it is a monomial. For $P \in \mathcal{P}_{\mathbb{L}}(x)$ there is a minimum $k_P \in \mathbb{N}$ such that $P(k_P) \in \mathbb{L}$. Then P is entirely determined by $P(0), \dots, P(k_P)$. We then define $\alpha_0(P), \dots, \alpha_{k_P}(P)$ as follows :

- Writing $x = \sum_{i<\nu(x)} r_i(x)\omega^{a_i(x)}$, then define $\alpha_0(P) < \nu(x)$ such that $P(0) = r_{\alpha_0(P)}(x)\omega^{a_{\alpha_0(P)}}$.
- For $0 \leq i < k$, write $P(i) = r\omega^a$. Then $P(i+1)$ is a term of $\ln \omega^a$. Write $\ln \omega^a = \sum_{i<\nu(a)} r_i(a)\omega^{h(a_i(a))}$. Then set $\alpha_{i+1}(P)$ such that $P(i+1) = r_{\alpha_{i+1}(P)}(a)\omega^{h(a_{\alpha_{i+1}(P)}(a))}$

Then P is entirely determined by the sequence $(\alpha_i(P))_{i\in[\![0\ ;\ k_P]\!]}$. This sequence satisfies

$$\forall i \in [\![0\ ;\ (k_P - 1)]\!] \qquad \alpha_{i+1}(P) < \nu(\ell(P(i)))$$

and

$$\alpha_0(P) < \nu(x) \leq \mathrm{NR}(x) \leq |x|_{+-}$$

Proposition 3.9.10. *Let $x \in \mathbf{No}$ and $P \in \mathcal{P}(x)$. Then for any $n \in \mathbb{N}$, the length of the series of $\ell(P(n))$, $\nu(\ell(P(n)))$ satisfies*

$$\nu(\ell(P(n))) \leq \mathrm{NR}(x) + 1$$

Proof. For any $x \in \mathbf{No}$ we write $x = \sum_{i<\nu(x)} r_i(x)\omega^{a_i(x)}$ in Gonshor's normal form. Now fix $x \in \mathbf{No}$. Let $P \in \mathcal{P}(x)$. We set $x_0 = x$, and $\alpha_0 < \nu(x)$ such $P(0) = r_{\alpha_0}(x)\omega^{a_{\alpha_0}(x_0)}$ and for any natural number n,

$$x_{n+1} = \ln \omega^{a_{\alpha_n}(x_n)} = \ell(P(n))$$

and

$$P(n+1) = r_{\alpha_{n+1}}\omega^{a_{\alpha_{n+1}}(x_{n+1})}$$

Using Proposition 3.8.14, we get $\qquad \mathrm{NR}(x_{n+1}) = \mathrm{NR}\left(\omega^{a_{\alpha_n}(x_n)}\right)$

By definition x_{n+1} is purely infinite. Then $a_{\alpha_{n+1}}(x_{n+1}) > 0$ for all natural number n. Since P is path, $P(0) \notin \mathbb{R}$ (otherwise $P(1)$ is not defined) and then $a_{\alpha_0}(x_0) \neq 0$. We then can apply Proposition 3.8.22 and get for all natural number n

$$\mathrm{NR}(x_{n+1}) \leq \mathrm{NR}\left(r_{\alpha_n}(x_n)\omega^{a_{\alpha_n}(x_n)}\right)$$

Now using Proposition 3.8.23, $\qquad \mathrm{NR}(x_{n+1}) \leq \mathrm{NR}(x_n)$

Then for any natural number n we have $\mathrm{NR}(x_n) \leq \mathrm{NR}(x_0) = \mathrm{NR}(x)$. Applying Lemma 3.8.20, we get

$$\forall n \in \mathbb{N} \qquad \nu(x_n) \leq \mathrm{NR}(x_n) + 1 \leq \mathrm{NR}(x) + 1$$

□

Remark 3.9.11. Actually, we often have $\nu(\ell(P(n))) \leq \mathrm{NR}(x)$. Indeed, using the notations of the proof and assuming that $\nu(x_{n+1}) = \mathrm{NR}(x) + 1$, we have

$$\mathrm{NR}(x) + 1 = \nu(x_{n+1}) \underset{\text{Proposition 3.9.10}}{\leq} \mathrm{NR}(x_{n+1}) + 1 \leq \dots \leq \mathrm{NR}(x) + 1$$

Then, all the inequalities are equalities and from Proposition 3.9.10 we get that x_{n+1} is a finite sum of terms of the form $\pm\mathbb{L}^{\pm 1}$, in particular $\nu(x_{n+1}) < \omega$ and $\mathrm{NR}(x)$ is finite.

Using Proposition 3.9.10, we get that $(\alpha_i(P))_{i\in[\![0\ ;\ k_P]\!]}$ is a finite sequence over ordinal less than $\mathrm{NR}(x)+1$. In particular, we can give $\mathcal{P}_{\mathbb{L}}(x)$ a lexicographic order inherited from the one over finite sequences.

Definition 3.9.12. We define the order $<_{lex}$ on paths by

$$P <_{lex} Q \iff (\alpha_0(P), \dots, \alpha_{k_P}(P)) <_{lex} (\alpha_0(Q), \dots, \alpha_{k_Q}(Q))$$

Lemma 3.9.13 ([12, Berarducci and Mantova, Corollary 6.17])**.** *Let $P, Q \in \mathcal{P}(x)$ such that $\partial P, \partial Q \neq 0$. If there is $i \in \mathbb{N}$ such that*

1. *$\forall j \leq i \qquad P(i) \preceq Q(i)$*
2. *$P(i+1)$ is not a term of $\ell(Q(i))$,*

then

$$\partial P \prec \partial Q$$

Lemma 3.9.14 ([12, Berarducci and Mantova, Lemma 6.18])**.** *Given $P \in \mathcal{P}(x)$ a path of x we have for all i*

$$\mathrm{NR}(P(i+1)) \leq \mathrm{NR}(P(i))$$

with equality if and only if $P(i)$ is the last term of $\ell(P(i))$. We also have

$$\mathrm{NR}(P(0)) \leq \mathrm{NR}(x)$$

with equality if and only if $P(0)$ is the last term of x.

We can define an ordering on path of $\mathcal{P}_{\mathbb{L}}(x)$:

Definition 3.9.15. On $\mathcal{P}_{\mathbb{L}}(x)$, we define the ordering :

$$P <_{\mathcal{P}} Q \Leftrightarrow (\partial P \succ \partial Q) \vee (\partial P \asymp \partial Q \wedge \partial P > \partial Q) \vee (\partial P = \partial Q \wedge P <_{lex} Q)$$

We now can state the theorem that builds a derivation from a prederivation.

Theorem 3.9.16 ([12, Berarducci and Mantova, Proposition 6.20, Theorem 6.32])**.** *Let $D_{\mathbb{L}}$ be a prederivation over a surreal field $\mathbb{K}$ stable under* $\exp$ *and* $\ln$. *Then $D_{\mathbb{L}}$ extends to a derivation $\partial : \mathbb{K} \to \mathbf{No}$ such that*

$$\forall x \in \mathbb{K} \qquad \partial x = \sum_{P \in \mathcal{P}(x)} \partial P$$

In particular, $\{\partial P\}_{P \in \mathcal{P}(x)}$ is summable (see Definition 3.9.1).

We will not go into the detail of the proof but we indicate that the proof strongly relies on Lemma 3.9.13.
In the previous proposition, there is a path derivative that has more importance than all the other ones. In particular, although some path derivatives can be combined and may not appear as an individual term in ∂x, there is one that is always alone in its equivalence class for $\asymp$, the path derivative of the dominant path.

Lemma 3.9.17 ([12, Berarducci and Mantova, Lemma 6.23])**.** *Let $x \in \mathbf{No} \setminus \mathbb{R}$ and P be the dominant path of x. Then $\partial P \neq 0$ and $\partial x \sim \partial P$.*

3.9.3 Berarducci and Mantova's simplest derivation

In [12], Berarducci and Mantova not only show how to build a derivation over surreal numbers, they show that among all the possible derivations, there is one that is simpler that all the others. We first introduce this derivation.

Definition 3.9.18 ([12, Berarducci and Mantova, Definition 6.7])**.** We define $\partial_{\mathbb{L}} : \mathbb{L} \to \mathbf{No}$ by

$$\forall \lambda \in \mathbb{L} \qquad \partial_{\mathbb{L}} \lambda = \exp\left(- \sum_{\alpha \in \mathbf{Ord} | \kappa_{-\alpha} \succeq^K \lambda} \sum_{n=1}^{+\infty} \ln_n \kappa_\alpha + \sum_{n=1}^{+\infty} \ln_n \lambda \right)$$

Example 3.9.19.

$$\partial_{\mathbb{L}} \omega = 1 \qquad \partial_{\mathbb{L}} \exp \omega = \exp \omega$$

$$\partial_{\mathbb{L}} \ln \omega = \exp(-\ln \omega) = \frac{1}{\omega} \qquad \partial_{\mathbb{L}} \ln_n \omega = \frac{1}{\prod_{k=0}^{n-1} \ln_k \omega}$$

$$\partial_{\mathbb{L}} \kappa_1 = \partial_{\mathbb{L}} \varepsilon_0 = \exp\left(\sum_{n=1}^{+\infty} \ln_n \kappa_1 \right) \qquad \partial_{\mathbb{L}} \kappa_{-1} = \exp\left(-\sum_{n=1}^{+\infty} \ln_n \omega \right)$$

In fact, κ_1 is intuitively $\exp_\omega \omega$. Therefore it is also quite intuitive that $\partial_{\mathbb{L}} \kappa_1 = \kappa_1 \ln(\kappa_1) \ln\ln(\kappa_1) \cdots$. The same happens for κ_{-1} which is intuitively $\ln_\omega \omega$. We indeed have $\partial_{\mathbb{L}} \kappa_{-1} = \dfrac{1}{\omega \ln(\omega) \ln\ln(\omega) \cdots}$.

Remark 3.9.20. For all $\lambda \in \mathbb{L}$, $\partial_{\mathbb{L}} \lambda$ is a positive monomial.

Proposition 3.9.21 ([12, Berarducci and Mantova, Propositions 6.9 and 6.10])**.** *$\partial_{\mathbb{L}}$ is a prederivation.*

The previous proposition ensures that the associated function ∂ defined by Theorem 3.9.16 is indeed a derivation over surreal numbers. It turns out that it the simplest for the order $\sqsubseteq$.

Remark 3.9.22. As mentioned by Berarducci and Mantova [12, Definition 6.6], we can define an other prederivation as follows:

$$\partial'_{\mathbb{L}} \lambda = \exp\left(\sum_{n=1}^{+\infty} \ln_n \lambda \right)$$

Although this definition seems to simpler than the one of $\partial_{\mathbb{L}}$, it turns out that this is just an illusion. Moreover, using this definition, we need to give up nice properties such that $\partial \omega = 1$. Indeed, we have $\partial'_{\mathbb{L}} \omega = \exp\left(\sum_{n=1}^{+\infty} \omega \right) = \dfrac{1}{\partial \kappa_{-1}}$. In fact, there is even no element x such that the associated derivation satisfies $\partial' x = 1$.

We now explain what is meant when saying that ∂ is the simplest derivation. In fact, we mean that $\partial_{\mathbb{L}}$ is the simplest prederivation with respect to the order $\sqsubseteq$.

Theorem 3.9.23 (Berarducci and Mantova[12, Theorem 9.6])**.** *Let D_λ be a prederivation. Let $\lambda \in \mathbb{L}$, minimal (in $\mathbb{L}$) for $\sqsubseteq$ such that $D_{\mathbb{L}} \lambda \neq \partial_{\mathbb{L}} \lambda$. Then $\partial_{\mathbb{L}} \lambda \sqsubset D_{\mathbb{L}} \lambda$.*

3.9.4 Objections to derivation

The derivation we introduced in the previous section may seems to be almost perfectly satisfying because of its simplicity property. Moreover it leads to almost all the properties one would expect from a formal calculus point of view. However, this definition of the derivation fails to fit any conceivable definition of the composition, *i.e* satisfying the chain rule. To show that we first specify what is expected from a composition.

Notation. Let us denote $\mathbf{No}^{>\mathbb{R}} = \{x \in \mathbf{No} \mid x > \mathbb{R}\}$.

Definition 3.9.24 (Berarducci and Mantova, [13, Definition 6.1])**.** A composition over the surreal number is a function $\circ : \mathbf{No} \times \mathbf{No}^{>\mathbb{R}}$ satisfying the following axioms:

Comp1 For a all summable family $(f_i)_{i\in I} \subseteq \mathbf{No}$ and $x \in \mathbf{No}^{>\mathbb{R}}$,

$$\left(\sum_{i\in I} f_i\right) \circ f = \sum_{i\in I} f_i \circ x$$

Comp2 For all $x \in \mathbf{No}^{>\mathbb{R}}$ and $r \in \mathbb{R}$, $$r \circ x = r$$

Comp3 For all $f \in \mathbf{No}$ and $x \in \mathbf{No}^{>\mathbb{R}}$, $$(\ln f) \circ x = \ln(f \circ x)$$

Comp4 For all $f \in \mathbf{No}$ and $g, x \in \mathbf{No}^{>\mathbb{R}}$, $$(f \circ g) \circ x = f \circ (g \circ x)$$
In particular, for $g, x \in \mathbf{No}^{>\mathbb{R}}$, $g \circ x \in \mathbf{No}^{>\mathbb{R}}$.

Comp5 For all $f \in \mathbf{No}$, $$f \circ \omega = f$$
and for all $f \in \mathbf{No}^{>\mathbb{R}}$, $$\omega \circ f = f$$

Remark 3.9.25. We could have defined $\circ$ over $\mathbb{K}$ a surreal field stable under $\exp$ and $\ln$ instead of **No**, just requiring an extra axiom of stability.

Remark 3.9.26. The point of Axioms **Comp1**, **Comp2** and **Comp3** understood with Axiom **Comp5** is to say that we just "substitute" the occurrences of ω in f by occurrences of x.

Now that we have a better understanding of what must be a composition, we show that it cannot fit the definition of ∂, the simplest derivative of Berarducci and Mantova.

Proposition 3.9.27 (Berarducci and Mantova, [13, Theorem 8.4])**.** *The derivative ∂ is not compatible (i.e the chain rule does not apply) with any composition over the surreal numbers. It is neither compatible if we restrict the composition to any surreal field containing at least κ_{-1} and some $\lambda \in \mathbb{L}$ such that $\lambda \succ^K \omega$.*

Proof. Assume that there is a composition $\circ$ such that the chain rule is valid for $\circ$ and ∂. Let $\lambda \in \mathbb{L}$. Recall that

$$\partial\kappa_{-1} = \exp\left(-\sum_{n=1}^{+\infty} \ln_n \omega\right)$$

Then
$$\begin{aligned}
\partial(\kappa_{-1} \circ \lambda) &= (\partial\lambda)((\partial\kappa_{-1}) \circ \lambda) && \text{(Chain rule)}\\
&= \exp\left(-\sum_{\alpha\in\mathbf{Ord}\mid\kappa_{-\alpha}\succeq^K\lambda}\sum_{n=1}^{+\infty} \ln_n \kappa_\alpha + \sum_{n=1}^{+\infty} \ln_n \lambda\right)((\partial\kappa_{-1}) \circ \lambda) && \text{(Definition 3.9.18)}\\
&= \exp\left(-\sum_{\alpha\in\mathbf{Ord}\mid\kappa_{-\alpha}\succeq^K\lambda}\sum_{n=1}^{+\infty} \ln_n \kappa_\alpha + \sum_{n=1}^{+\infty} \ln_n \lambda\right)\exp\left(-\sum_{n=1}^{+\infty} \ln_n \lambda\right) && (\mathbf{Comp1}, \mathbf{Comp3}, \mathbf{Comp5})\\
&= \exp\left(-\sum_{\alpha\in\mathbf{Ord}\mid\kappa_{-\alpha}\succeq^K\lambda}\sum_{n=1}^{+\infty} \ln_n \kappa_\alpha\right)
\end{aligned}$$

In particular, for any $\lambda \succ^K \omega$, $$\partial(\kappa_{-1} \circ \lambda) = 1$$ which contradicts Proposition 3.9.4. □

Remark 3.9.28. Notice that the fact κ_{-1} must belong to the field which we are working on is not huge assumption. Indeed, if we want it to be stable under $\exp$, $\ln$, ∂ and also have access to primitives, we definitively want $-\sum_{n=1}^{+\infty} \ln_n \omega$ to belong to our field. Therefore κ_{-1} is also in the field as the primitive of $\exp\left(-\sum_{n=1}^{+\infty} \ln_n \omega\right)$. The existence of a log-atomic $\lambda \succ^K \omega$ such as κ_1 is also quite natural if we want also want to have access to very fast increasing growth rate.

3.9.5 Some bounds about the derivative

In this section we give some bounds about $\nu(\partial x)$ and $\mathrm{NR}(\partial x)$ for x a surreal number. It will be very useful to get surreal fields stable under the derivation and the anti-derivation.

Proposition 3.9.29. *For any $x \in \mathbf{No}$, the set $\mathcal{P}_{\mathbb{L}}(x)$ is well-ordered with order type $\beta < \omega^{\omega^{\omega(\mathrm{NR}(x)+1)}}$. In particular,*

$$\nu(\partial x) < \omega^{\omega^{\omega(\mathrm{NR}(x)+1)}}$$

Proof. We know that $\{\partial P\}_{P\in\mathcal{P}(x)}$ is summable (see Definition 3.9.1). In particular $\{\partial P\}_{P\in\mathcal{P}_{\mathbb{L}}(x)}$ is summable. By definition of summability (in this context) for any $P \in \mathcal{P}_{\mathbb{L}}(x)$, there are finitely many $Q \in \mathcal{P}_{\mathbb{L}}(x)$ such that $\partial P \asymp \partial Q$. By definition of summability, $<_{\mathcal{P}}$ is a well total order over $\mathcal{P}_{\mathbb{L}}(x)$ and if β is its order type, then $\omega \otimes \nu(\partial x) < \beta$ (usual ordinal product). Then, to complete the proof, we just need to show that $\beta < \omega^{\omega^{\omega(\mathrm{NR}(x)+1)}}$. We proceed by induction on $\mathrm{NR}(x)$.

- $\underline{\mathrm{NR}(x)=0}$: then $x = 0$ or $x = \pm y^{\pm 1}$ for some $y \in \mathbb{L}$ and $\nu(\partial x) \leq 1 < \omega^{\omega^{\omega}}$ and we conclude the proof.
- Assume that for any y such that $\mathrm{NR}(y) < \mathrm{NR}(x)$, $\mathcal{P}_{\mathbb{L}}(y)$ has order type less than $\omega^{\omega^{\omega(\mathrm{NR}(y)+1)}}$. Assume for contradiction that $\beta \geq \omega^{\omega^{\omega(\mathrm{NR}(x)+1)}}$. Then for any multiplicative ordinal $\mu < \omega^{\omega^{\omega(\mathrm{NR}(x)+1)}}$, there is some $P_\mu \in \mathcal{P}_{\mathbb{L}}(x)$, minimum with respect to $<_{lex}$, such that the set

$$\mathcal{E}_\mu(x) = \{\, Q \in \mathcal{P}_{\mathbb{L}}(x) \mid Q <_{\mathcal{P}} P_\mu \}$$

has order type $\beta_\mu \geq \mu$. Let us select any μ such that $\mu \geq \omega^{\omega^{\omega\,\mathrm{NR}(x)+1}}$. Now define

$$\mathcal{E}^{(1)}_\mu(x) = \{\, Q \in \mathcal{P}_{\mathbb{L}}(x) \mid Q <_{\mathcal{P}} P_\mu \quad Q <_{lex} P_\mu \}$$

$$\mathcal{E}^{(2)}_\mu = \{\, Q \in \mathcal{P}_{\mathbb{L}}(x) \mid Q >_{lex} P_\mu \}$$

Theses sets are disjoints and
$$\mathcal{E}_\mu = \mathcal{E}^{(1)}_\mu \cup \mathcal{E}^{(2)}_\mu$$

Let $\beta^{(i)}_\mu$ be the order type of $\mathcal{E}^{(i)}_\mu$. We then have

$$\mu \leq \beta_\mu \leq \beta^{(1)}_\mu + \beta^{(2)}_\mu$$

where the addition is the surreal addition of ordinal numbers. Since μ is multiplicative ordinal, hence, an additive one, at least one of the $\beta^{(i)}_\mu \geq \mu$.

➢ $\underline{\text{First case}}$: $\beta^{(2)}_\mu \geq \mu$. Since μ is additive, there is an $i \in [\![0\,;\, k_P]\!]$ such that the well ordered set

$$\mathcal{E}^{(2,i)}_\mu = \left\{ Q \in \mathcal{E}^{(2)}_\mu \;\middle|\; \forall j < i\; Q(j) = P_\mu(j) \quad Q(i) \prec P_\mu(i) \right\}$$

has order type at least μ. We take such an i. For $Q \in \mathcal{E}^{(2,i)}_\mu$, we consider the path $Q'(n) = Q(n+i+1)$. Since $\partial Q \succeq \partial P_\mu$, Lemma 3.9.13 gives us that $Q(i+1)$ is a term of $\ell(P_\mu(i))$. We then have $Q' \in \mathcal{P}\left(\ell(P_\mu(i))\right)$ and

$$\partial Q' = \frac{\partial Q}{Q(0)\cdots Q(i)} = \frac{\partial Q}{P_\mu(0)\cdots P_\mu(i-1)Q(i)}$$

In particular $Q' \in \mathcal{P}_{\mathbb{L}}\left(\ell(P_\mu(i))\right)$. Since $Q(i) \prec P_\mu(i)$, $P_\mu(i)$ is not the last term of $\ell(P_\mu(i-1))$ (or x if $i=0$). Then Proposition 3.8.23 ensures that

$$\mathrm{NR}(\ell(P_\mu(i))) \leq \mathrm{NR}(P_\mu(i)) < \mathrm{NR}(x)$$

Applying the induction hypothesis on $\ell(P_\mu(i))$, the order type of $\mathcal{P}_{\mathbb{L}}((\ell(P_\mu(i))))$ has order type γ such that

$$\gamma < \omega^{\omega^{\omega(\mathrm{NR}(\ell(P(i)))+1)}} \leq \omega^{\omega^{\omega\,\mathrm{NR}(x)}} < \omega^{\omega^{\omega\,\mathrm{NR}(x)+1}} \leq \mu$$

For $Q, R \in \mathcal{E}^{(2,i)}_\mu$, $Q <_{\mathcal{P}} R$ *iff*

$(Q(i)\partial Q' \succ R(i)\partial R') \vee (Q(i)\partial Q' \asymp R(i)\partial R' \wedge Q(i)\partial Q' > R(i)\partial R') \vee (Q(i)\partial Q' = R(i)\partial R \wedge Q <_{lex} R)$

what we can also write

$$\begin{aligned} Q <_{\mathcal{P}} R \Leftrightarrow\; & \big(\ell(Q(i)) + \ell(\partial Q') > \ell(R(i)) + \ell(\partial R')\big) \\ & \vee\big(\ell(Q(i)) + \ell(\partial Q') = \ell(R(i)) + \ell(\partial R') \wedge Q(i)\partial Q' > R(i)\partial R'\big) \\ & \vee(Q(i)\partial Q' = R(i)\partial R \wedge Q <_{lex} R) \end{aligned}$$

where the two later cases occur finitely may times for Q or R fixed. Let δ denote the order type of the possible values for $Q(i)$ and $\beta^{(2,i)}_\mu$ the order type of $\mathcal{E}^{(2,i)}_\mu$. Since ℓ is non-decreasing, $\left\{ \ell(\partial Q') \;\middle|\; Q \in \mathcal{E}^{(2,i)}_\mu \right\}$ has order type at most γ and $\left\{ \ell(Q(i)) \;\middle|\; Q \in \mathcal{E}^{(2,i)}_\mu \right\}$ has order type at most $\mathrm{NR}(x)$. Using Proposition 2.4.3,

$$\beta^{(2,i)}_\mu \leq (\gamma\,\mathrm{NR}(x)) \otimes \omega < \mu$$

Finally
$$\mu \leq \beta^{(2,i)}_\mu < \mu$$
and we reach the contradiction.

➢ Second case : $\beta_\mu^{(2)} < \mu$. Then $\beta_\mu^{(1)} \geq \mu$. Let us define for $i \in [\![0 ; k_P]\!]$

$$\mathcal{E}_\mu^{(1,i)} = \left\{ Q \in \mathcal{E}_\mu^{(1)} \;\middle|\; \forall j < i \; P_\mu(j) = Q(j) \quad P_\mu(i) \prec Q(i) \right\}$$

Since there are finitely many of them, that they form a partition of $\mathcal{E}_\mu^{(1)}$ and μ is multiplicative, hence additive, there is at least one of them which has order type at least μ. We consider such an $i \in [\![0 ; k_P]\!]$. Now define

$$x_j = \begin{cases} x & i = j \\ \ell(P(i-j-1)) & j < i \end{cases}$$

Writing $x_0 = \sum_{n<\nu(x_0)} r_n(x_0)\omega^{a_n(x_0)}$ and $P_\mu(i) = r_{\alpha_0}(x_0)\omega^{a_{\alpha_0}(x_0)}$ we set

$$y_0 = \sum_{n<\alpha_0} r_n(x_0)\omega^{a_n(x_0)}$$

Now for $0 \leq j < i$, we define y_{j+1} has follows. We have that $P_\mu(i-j-1)$ is a term of x_{j+1}. Write $P_\mu(i-j-1) = r_{\alpha_{j+1}}(x_{j+1})\omega^{a_{\alpha_{j+1}}(x_{j+1})}$ for some $\alpha_{j+1} < \nu(x_{j+1})$. Then set

$$y_{j+1} = \sum_{n<\alpha_{j+1}} r_n(x_{j+1})\omega^{a_n(x_{j+1})} + \operatorname{sign}(r_{\alpha_{j+1}}(x_{j+1}))\exp(y_j)$$

Denote $y = y_i$. For $Q \in \mathcal{E}_\mu^{(1,i)}$. For any $Q \in \mathcal{E}_\mu^{(1,i)}$ we will build $Q' \in \mathcal{P}_\mathbb{L}(y)$. We expect to use the induction hypothesis on y. First we prove that $\mathrm{NR}(y) < \mathrm{NR}(x)$. In fact, by trivial induction, we have $y_j \lhd_j x_j$. So $y \lhd_i x$ and by definition of NR we have $\mathrm{NR}(y) < \mathrm{NR}(x)$. Now consider the path Q' defined as follows :

$\because \; \forall j < i \qquad Q'(j) = \operatorname{sign}(r_{\alpha_j}(x_{i-j}))\exp(y_{i-j-1})$

$\because \; \forall j \geq i \qquad Q'(j) = Q(j)$

We then have $Q' \in \mathcal{P}(y)$. We can even say $Q' \in \mathcal{P}_\mathbb{L}(y)$. Moreover, since we change only the common terms of the path, and the changes do not depend on Q, we have

$$\forall Q, R \in \mathcal{E}_\mu^{(1,i)} \qquad Q <_\mathcal{P} R \Leftrightarrow Q' <_\mathcal{P} R'$$

We then have an increasing function

$$\Phi : \begin{cases} \mathcal{E}_\mu^{(1,i)} & \to & \mathcal{P}_\mathbb{L}(y) \\ Q & \mapsto & Q' \end{cases}$$

The induction hypothesis give that the order type of $P'(y)$ is less than $\omega^{\omega^{\omega(\mathrm{NR}(y)+1)}}$. Then

$$\omega^{\omega^{\omega\,\mathrm{NR}(x)}} \leq \mu < \omega^{\omega^{\omega(\mathrm{NR}(y)+1)}} \leq \omega^{\omega^{\omega\,\mathrm{NR}(x)}}$$

and we get the contradiction.

This completes the proof.

□

Corollary 3.9.30. *Let λ be an ε-number. If $\mathrm{NR}(x) < \lambda$ then $\nu(\partial x) < \lambda$*

Proposition 3.9.31. *For all $x \in \mathbf{No}$, let α the minimum ordinal such that $\kappa_{-\alpha} \prec^K t$ for all log-atomic t such that there is some path $P \in \mathcal{P}_\mathbb{L}(x)$ and some index $k \in \mathbb{N}$ such that $P(k) = t$. Then, for all path P,*

$$\mathrm{NR}(\partial P) \leq k(\mathrm{NR}(x)+1) + \omega(\alpha+1)$$

and

$$\mathrm{NR}(\partial x) \leq \omega(\mathrm{NR}(x)+\alpha+2) \otimes \nu(\partial x) \leq \omega^{\omega^{\omega(\mathrm{NR}(x)+1)}+\alpha}$$

Proof. Let P be a path of such that $\partial P \neq 0$. Then there is some $k \in \mathbb{N}$ such that $\partial P = P(0) \cdots P(k-1)\partial_\mathbb{L} P(k)$. With Corollary 3.8.25, we get

$$\begin{aligned}
\mathrm{NR}(\partial P) &\leq \sum_{i=0}^{k-1} \mathrm{NR}(P(i)) + \mathrm{NR}(\partial_\mathbb{L} P(k)) + k \\
&\leq k\,\mathrm{NR}(x) + \mathrm{NR}(\partial_\mathbb{L} P(k)) + k && \text{(Lemma 3.9.14)} \\
&\leq k\,\mathrm{NR}(x) + k + \mathrm{NR}\left(\exp\left(-\sum_{\kappa_{-\beta} \succeq^K P(k)} \sum_{n\geq 1} \ln_n \kappa_{-\beta} + \sum_{n\geq 1} \ln_n P(k)\right)\right) \\
&\leq k\,\mathrm{NR}(x) + k + \mathrm{NR}\left(-\sum_{\kappa_{-\beta} \succeq^K P(k)} \sum_{n\geq 1} \ln_n \kappa_{-\beta} + \sum_{n\geq 1} \ln_n P(k)\right) && \text{(Proposition 3.8.14)} \\
&\leq k\,\mathrm{NR}(x) + k + (\omega \otimes (\alpha \oplus 1)) && \text{(Lemma 3.8.19)} \\
&\leq \omega(\mathrm{NR}(x)+1) + \omega(\alpha+1)
\end{aligned}$$

This bound does not depend on P. Then applying Proposition 3.9.29 and Lemma 3.8.18 we get

$$\mathrm{NR}(\partial x) \leq (\omega(\mathrm{NR}(x)+\alpha+2)) \otimes \nu(\partial x) < (\omega(\mathrm{NR}(x)+\alpha+2)) \otimes \omega^{\omega^{\omega(\mathrm{NR}(x)+1)}} \leq \omega^{\omega^{\omega(\mathrm{NR}(x)+1)}+\alpha}$$

□

3.10 Anti-derivation

In this section, we mainly follow [12, Section 7] to conclude the same way. However, we may have some more difficulties than Berarducci and Mantova since we won't allow us to use classes theory so that we can actually consider the proof in any surreal field. Indeed, with actual fields we cannot exhibit a proper class as they do in [12] to prove the existence of a primitive. However we still first consider an asymptotic anti-derivative and then we will build an accurate one. This work takes its origin in the work of Rosenlicht [39] on Hardy fields, Kuhlmann and Matusinki [31, 32] on Hardy type derivations for transseries and power series.

3.10.1 Asymptotic anti-derivation

We recall the following theorem first used by Berarducci and Mantova [12] to exhibit an asymptotic anti-derivation.

Theorem 3.10.1 ([39, Rosenlicht, Theorem 1]). *Let $\mathbb{K}$ be a Hardy field with valuation v. Let*

$$\Psi = \left\{ \frac{f'}{f} \;\middle|\; f \in \mathbb{K} \quad v(f) \neq 0 \right\}$$

For all $f \in \mathbb{K}^*$, if $v(f) \neq \inf \Psi$, then there is $u_0 \in \mathbb{K}^*$ with $v(u_0) \neq 0$ such that for all $u \in \mathbb{K}^*$ such that $0 < |v(u)| \leq |v(u_0)|$,

$$f \sim \left(f \frac{\frac{fu}{u'}}{\left(\frac{fu}{u'}\right)'} \right)'$$

Recall that a valuation v over $\mathbb{K}$ is a function defined of $\mathbb{K}^*$ such that

- $\forall x, y \in \mathbb{K} \qquad x + y \neq 0 \Rightarrow v(x+y) \geq \min(v(x), v(y))$
- $\forall x, y \in \mathbb{K} \qquad v(xy) = x(x) + v(y)$

It is easy to see that $-\ell$ is a valuation on **No** and thus on any surreal field $\mathbb{K}$.

Proposition 3.10.2 ([12, Berarducci and Mantova, Proposition 7.2]). *The set $\Psi_{\mathbb{L}} = \left\{ \ell\left(\frac{\partial x}{x}\right) \;\middle|\; x \in \mathbb{L} \right\}$ has no infimum in the class of purely infinite numbers.*

Corollary 3.10.3 ([12, Berarducci and Mantova, Corollary 7.3]). *The set $\Psi = \left\{ \ell\left(\frac{\partial x}{x}\right) \;\middle|\; x \in \mathbf{No} \quad \ell(x) \neq 0 \right\}$ has no infimum in the class of purely infinite numbers.*

Proposition 3.10.4 ([12, Berarducci and Mantova, Proposition 7.4]). *There is a class function $A : \mathbf{No}^* \to \mathbb{R}\omega^{\mathbf{No}^*}$ such that*

$$\forall x \in \mathbf{No}^* \qquad x \sim \partial A(x)$$

The above proposition is namely an application of Theorem 3.10.1. Basically, $A(x)$ is the leading term of $x \dfrac{xu/\partial u}{\partial(xu/\partial u)}$ where $u = \kappa_\alpha$ and α minimal such that it satisfies the condition in Theorem 3.10.1. Actually we can be even more precise.

Proposition 3.10.5. *Let $x \in \mathbf{No}^*_+$ such that $\ln x \not\sim -\ln\omega$ and t be its leading term. Then*

$$A(x) = \begin{cases} \dfrac{t^2}{\partial P} & \text{if } \ln x \succ \ln\omega \quad \text{and} \quad P \text{ is the dominant path of } x \\ \dfrac{\omega t}{r+1} & \text{if } \ln x \sim r\ln\omega \quad \text{and} \quad r \in \mathbb{R} \setminus \{0, -1\} \\ \omega x & \text{if } \ln x \prec \ln\omega \end{cases}$$

The proof is really straightforward. We just have to check that $\partial A(x) \sim x$ by computing $\partial A(x)$. The following proof is based on Rosenlicht's theorem 3.10.1 to show how to get the expression from the theorem.

Proof. Let u_0 given by Theorem 3.10.1 and $\alpha \geq 1$ minimal such that $0 < \ell(\kappa_{-\alpha}) \leq |\ell(u_0)|$. Let also

$$y = x \frac{x\kappa_{-\alpha}/\partial\kappa_{-\alpha}}{\partial(x\kappa_{-\alpha}/\partial\kappa_{-\alpha})}$$

Then $\partial y \sim x$. Let us re-write

$$y = \frac{x}{\partial \ln\left(x\kappa_{-\alpha}/\partial\kappa_{-\alpha}\right)} = \frac{x}{\partial\ln x + \partial\ln\kappa_{-\alpha} - \partial\ln\partial\kappa_{-\alpha}}$$

Since $\kappa_{-\alpha}$ is log-atomic we then have

$$\begin{aligned}
\partial \ln \kappa_{-\alpha} &= \exp\left(-\sum_{\beta<\alpha}\sum_{n\in\mathbb{N}^*} \ln_n \kappa_{-\beta} - \ln \kappa_{-\alpha}\right)\\
&= \exp\left(-\sum_{\beta<\alpha}\sum_{n\in\mathbb{N}^*} \omega^{\omega^{-\omega\otimes\beta-n}}\right)\omega^{-\omega^{-\omega\otimes\alpha}}\\
&= \omega^{-\sum_{\beta<\alpha}\sum_{n\in\mathbb{N}} \omega^{-\omega\otimes\beta-n}-\omega^{-\omega\otimes\alpha}}\\
&= \underbrace{\omega^{-\sum_{n\in\mathbb{N}}\omega^{-n}}}_{\prec\omega^{-1}} \times \underbrace{\omega^{-\sum_{1\leq\beta<\alpha}\sum_{n\in\mathbb{N}}\omega^{-\omega\otimes\beta-n}-\omega^{-\omega\otimes\alpha}}}_{\preceq 1}\\
\partial \ln \kappa_{-\alpha} &\prec \frac{1}{\omega}
\end{aligned}$$

and

$$\begin{aligned}
\partial \ln \partial\kappa_{-\alpha} &= \partial\left(-\sum_{\beta<\alpha}\sum_{n\in\mathbb{N}^*} \ln_n \kappa_{-\beta}\right)\\
&= -\sum_{\beta<\alpha}\sum_{n\in\mathbb{N}^*} \partial \ln_n \kappa_{-\beta}\\
&\sim -\frac{1}{\omega}
\end{aligned}$$

We can now split into cases

- $\ln x \prec \ln \omega$: Then, since $\ln \omega \not\asymp 1$, from Proposition 3.9.4, we get

  $$\partial \ln x \prec \partial \ln \omega = 1/\omega \sim (\partial \ln \kappa_{-\alpha} - \partial \ln \partial \kappa_{-\alpha})$$

  and
  $$y \sim \frac{x}{\partial \ln \omega} = \omega x$$

  Moreover, if $x \sim r\frac{1}{\omega}$ for some $r \in \mathbb{R}_+^*$, then $\ln x = -\ln \omega + \ln r + \ln(1+\varepsilon)$ for some infinitesimal ε. Then $\ln x \asymp \ln \omega$ what is not. We can apply again Proposition 3.9.4 and get $x \sim \partial y \sim \partial(\omega x) \sim \partial(\omega t)$.

- $\ln x \sim r \ln \omega$ for some $r \neq 0, -1$. Then, using the same argument as above,

  $$y \sim \frac{x}{(r+1)\partial \ln \omega} = \frac{\omega x}{r+1}$$

  and we conclude the same way.

- $\ln x \succ \ln \omega$. Then again we can conclude

  $$y \sim \frac{x}{\partial \ln x} = \frac{x^2}{\partial x} \underset{\text{Lemma 3.9.17}}{\sim} \frac{t^2}{\partial P}$$

□

We now generalize the previous proposition to manage all cases. We focus on the special case of x of the form $x = \partial u \exp(\varepsilon u)$ and $\varepsilon \prec 1$. The intuition is that $u = \ln_n \kappa_{-\alpha}$ for some $n \in \mathbb{N}$ and some ordinal α and in analogous cases of the previous proposition, we get a the form of the anti-derivative.

Lemma 3.10.6. *Let $u = \ln_n \kappa_{-\alpha}$ for some $n \in \mathbb{N}$ and some ordinal α. Let $x = \partial u \exp \varepsilon$. If $\varepsilon \succ \ln u$, then*

$$\partial\left(\frac{x}{\partial\varepsilon}\right) \sim x$$

Proof. Let $y = \frac{x}{\partial\varepsilon} = \frac{\partial u}{\partial\varepsilon}\exp(\varepsilon)$. Since $\varepsilon \succ \ln u$, Proposition 3.9.4 ensures that $\partial\varepsilon \succ \frac{\partial u}{u}$. Then, $\frac{\partial u}{\partial \varepsilon} \prec u \not\asymp 1$

$$\begin{aligned}
\partial y &= \frac{\partial u}{\partial\varepsilon}\partial\varepsilon\exp(\varepsilon) + \partial\left(\frac{\partial u}{\partial\varepsilon}\right)\exp(\varepsilon)\\
&= x + \partial\left(\frac{\partial u}{\partial\varepsilon}\right)\exp(\varepsilon)
\end{aligned}$$

Proposition 3.9.4 gives that $\partial\left(\frac{\partial u}{\partial\varepsilon}\right) \prec \partial u$. Then $\quad \partial y \sim x$

□

Lemma 3.10.7. *Let $u = \ln_n \kappa_{-\alpha}$ for some $n \in \mathbb{N}$ and some ordinal α. Let $x = \partial u \exp(\varepsilon)$. If $\varepsilon \sim r \ln u$ for some $r \in \mathbb{R} \setminus \{0, -1\}$, then*

$$\partial\left(\frac{1}{r+1}\frac{ux}{\partial u}\right) \sim x$$

Proof. Let us compute the above derivative.

$$\partial\left(\frac{1}{r+1}\frac{ux}{\partial u}\right)=\partial\left(\frac{u\exp(\varepsilon)}{r+1}\right)=\frac{x}{r+1}+\frac{u\partial\varepsilon\exp(\varepsilon)}{r+1}$$

Using Proposition 3.9.4, we get that $\partial\varepsilon\sim\partial(r\ln u)=r\frac{\partial u}{u}$. Then, since $r\neq -1$, we get that

$$\partial\left(\frac{1}{r+1}\frac{ux}{\partial u}\right)\sim x$$

□

Lemma 3.10.8. *Let $u=\ln_n\kappa_{-\alpha}$ for some $n\in\mathbb{N}$ and some ordinal α. Let $x=\partial u\exp(\varepsilon)$. If $\varepsilon\prec\ln u$, then*

$$\partial\left(\frac{ux}{\partial u}\right)\sim x$$

Proof. Let us compute the above derivative.

$$\partial\left(\frac{ux}{\partial u}\right)=\partial\left(u\exp(\varepsilon)\right)=x+u\partial\varepsilon\exp(\varepsilon)$$

Using Proposition 3.9.4, we get that $\partial\varepsilon\prec\partial\ln u=\frac{\partial u}{u}$. Then, $u\partial\varepsilon\exp(\varepsilon)\prec x$ and we get that

$$\partial\left(\frac{ux}{\partial u}\right)\sim x$$

□

Theorem 3.10.9. *Let x be a term. Write $|x|=\partial u\exp(\varepsilon)$ with $u=\ln_n\kappa_{-\alpha}=\lambda_{-\omega\alpha-n}$ with $\omega\alpha+n$ such minimal that $\varepsilon\not\sim-\ln u$. Then,*

$$A(x)\sim\begin{cases}\dfrac{x}{\partial\varepsilon} & \varepsilon\succ\ln u\\ \dfrac{ux}{(r+1)\partial u} & \varepsilon\sim r\ln u\quad r\neq 0,-1\\ \dfrac{ux}{\partial u} & \varepsilon\prec\ln u\end{cases}$$

Proof. Since $A(x)=-A(-x)$, we may assume that $x>0$. Then, we just need to apply Lemmas 3.10.6, 3.10.7, and 3.10.8. □

Then we can give a more explicit formula for Berarducci and Mantova's asymptotic anti-derivation [12].

Corollary 3.10.10. *Let x be a non-zero surreal number. Write $|x|=\partial u\exp(\varepsilon)$ with $u=\ln_n\kappa_{-\alpha}=\lambda_{-\omega\alpha-n}$ with $\omega\alpha+n$ such minimal that $\varepsilon\not\sim-\ln u$. Then,*

$$A(x)=\begin{cases}\dfrac{t}{s} & \varepsilon\succ\ln u\\ \dfrac{ut}{(r+1)\partial u} & \varepsilon=r\ln u+\eta\quad r\neq -1,\eta\prec\ln u\end{cases}$$

where t is the leading term of x and s the leading term of $\partial\varepsilon$.

Proof. Just use Theorem 3.10.9 and the definition of A. □

This point of view will be very useful to determine when a surreal field is stable under anti-derivation.

3.10.2 General anti-derivation

We are now ready to build the anti-derivation for surreal numbers. We start with a useful lemma due to Aschenbrenner, van den Dries and van der Hoeven and we provide an other proof of it that fits our context. We also express it in way that matches our notations.

Definition 3.10.11. A function Φ is **strongly linear** is for all summable family $\{x_i\}_{i\in I}$,

$$\Phi\left(\sum_{i\in I}x_i\right)=\sum_{i\in I}\Phi(x_i)$$

Lemma 3.10.12 ([4, Aschenbrenner, van den Dries, van der Hoeven, Corollary 1.4])**.** *Let Φ a strongly linear map defined over a field $\mathbb{K}$ of surreal numbers. Assume that for any monomial $\omega^a\in\mathbb{K}$, we have $\Phi(\omega^a)\prec\omega^a$. Then $\sum_{n\in\mathbb{N}}\Phi^n(x)$ makes sense as a surreal number (i.e $\{\Phi^n(x)\}_{n\in\mathbb{N}}$ is summable) and if it belongs to $\mathbb{K}$ for all x, we have*

$$(\mathrm{id}-\Phi)^{-1}=\sum_{n\in\mathbb{N}}\Phi^n$$

Proof. Let $x \in \mathbb{K}$ be a surreal number and the sequence

$$\begin{cases} x_0 = x \\ x_{n+1} = \Phi(x_n) \end{cases}$$

To show that $\{x_n\}_{n\in\mathbb{N}}$ is summable, we need to show that $\bigcup\limits_{n\in\mathbb{N}} \operatorname{supp} x_n$ is reverse well-ordered and that for any $a \in \bigcup\limits_{n\in\mathbb{N}} \operatorname{supp} x_n$, the set $\{\, n \in \mathbb{N} \mid a \in \operatorname{supp} x_n \}$ is finite.

- Let us assume that $\bigcup\limits_{n\in\mathbb{N}} \operatorname{supp} x_n$ is not reverse well-ordered. Then there is an increasing infinite sequence $(a_n)_{n\in\mathbb{N}}$ in this set. Let $i_n \in \mathbb{N}$ such that $a_n \in \operatorname{supp} x_{i_n}$. Since for all k, $\bigcup\limits_{i<k} x_i$ is reverse well ordered, up to extraction we can assume i_n to be increasing. We then consider a specific sequence $(a_n)_{n\in\mathbb{N}}$ as follows :
  - ➢ Take i_0 minimum such that there is an increasing sequence $(a_n)_{n\in\mathbb{N}}$ with $a_0 \in \operatorname{supp} x_{i_0}$. Now take a_0 maximum in $\operatorname{supp} x_{i_0}$ with such a property.
  - ➢ Take $i_{n+1} > i_n$ minimum such that there is an increasing sequence $(b_n)_{n\in\mathbb{N}}$ with $b_{n+1} \in \operatorname{supp} x_{i_{n+1}}$ and $b_k = a_k$ for $k \leq n$. Take $a_{n+1} \in \operatorname{supp} x_{i_{n+1}}$ to be the maximum b_{n+1} with such a property.

  For $n \geq 1$ we take $b_n \in \operatorname{supp} x_{i_n - 1}$ such that $a_n \in \operatorname{supp} \Phi(\omega^{b_n})$. Such a quantity always exists otherwise $a_n \notin \operatorname{supp} x_{i_n}$. Assume $\{\, b_n \mid n \in \mathbb{N}^* \}$ is not reverse well-ordered. Take $(b_{j_n})_{n\in\mathbb{N}^*}$ be an increasing extraction of $(b_n)_{n\in\mathbb{N}^*}$. In particular, since $\Phi(\omega^a) \prec \omega^a$ for all a, we have $a_{j_n - 1} < a_{j_n} < b_{j_n}$. We now split into two cases :
  - ➢ If $i_{j_n} - 1 = i_{j_n - 1}$, then b_{j_n} contradicts the maximality of $a_{j_n - 1}$.
  - ➢ If not, then $i_{j_n} - 1$ contradicts the minimality of i_{j_n}.

  Then, we get a contradiction. We the form the surreal number $y = \sum\limits_{n\in\mathbb{N}^*} \omega^{b_n}$. Then $\operatorname{supp} \Phi(y)$ is reverse well-ordered and must contains all the a_ns (eventually except a_0), what is impossible. Then $\bigcup\limits_{n\in\mathbb{N}} \operatorname{supp} x_n$ is reverse well-ordered.

- Assume there is a such that $E_a := \{\, n \in \mathbb{N} \mid a \in \operatorname{supp} x_n \}$ is infinite. Choose a maximal for that property. For all $n \in E_a \backslash \{0\}$ take $b_n \in \operatorname{supp} x_{n-1}$ such that $a \in \Phi(\omega^{b_n})$. As a subset of $\bigcup\limits_{n\in\mathbb{N}} \operatorname{supp} x_n$, $\{\, b_n \mid n \in E_a \setminus \{0\}\}$ is reverse well-ordered. However the family $\left\{\omega^{b_n}\right\}_{n\in E_a\backslash\{0\}}$ must not be summable otherwise the family $\left\{\Phi(\omega^{b_n})\right\}_{n\in E_a\backslash\{0\}}$ must be also summable. It is not possible since by definition E_a is infinite and a appear in each support of terms of the sum. Then there is b such that $\{\, n \mid b_n = b\}$ is infinite. Since $\Phi(\omega^b) \prec \omega^b$, b contradicts the maximality of a. Finally, E_a is finite for all a.

Then $\sum\limits_{n\in\mathbb{N}} \Phi^i$ makes sens. Now if for all $x \in \mathbb{K}$, $\sum\limits_{n\in\mathbb{N}} \Phi^i(x) \in \mathbb{K}$, then, by strong linearity of Φ, we can see that

$$(\operatorname{id} - \Phi) \circ \sum_{n\in\mathbb{N}} \Phi^i = \operatorname{id}$$

□

Definition 3.10.13. We define an extension of A, denoted $\mathcal{A}$, to all surreal numbers by

$$\mathcal{A}\left(\sum_{i<\nu} r_i \omega^{a_i}\right) = \sum_{i<\nu} r_i A(\omega^{a_i})$$

We also introduce the function $\Phi = \operatorname{id} - \partial \circ \mathcal{A}$.

Proposition 3.9.4 ensures that the function $\mathcal{A}$ is well defined. Moreover, this function is obviously strongly linear. We now consider, given a surreal number x, the sequence

$$\begin{cases} x_0 = x \\ x_{n+1} = x_n - \partial\mathcal{A}(x_n) = \Phi(x_n) \end{cases}$$

Note that if $\omega^a = \partial u \exp \varepsilon$ with $u = \lambda_{-\omega\otimes\alpha - n} = \ln_n \kappa_{-\alpha}$ and $\varepsilon \prec \ln \lambda_{-\omega\otimes\beta - m}$ for $\omega\otimes\beta + m < \omega\otimes\alpha + n$, and $\omega\otimes\alpha + n$ maximum for that property, we have

$$\Phi(\omega^a) = \begin{cases} \left(1 - \dfrac{\partial\varepsilon}{s}\right)\omega^a - \partial\left(\dfrac{\partial u}{s}\right)\exp\varepsilon & \varepsilon \succ \ln u \quad s \text{ dominant term of } \partial\varepsilon \\ \dfrac{\omega^a}{r+1}\partial\eta\dfrac{u}{\partial u} & \varepsilon = r\ln u + \eta \quad r \neq -1 \end{cases}$$

Corollary 3.10.14. *The operator* $\operatorname{id} - \Phi$ *is invertible with inverse* $\sum\limits_{i\in\mathbb{N}} \Phi^i$*. Moreover* $\mathcal{A} \circ \sum\limits_{i\in\mathbb{N}} \Phi^i$ *is an operator that sends every* x *to some anti-derivative of* x*.*

Proof. Lemma 3.10.12 ensure that $\mathrm{id} - \Phi$ has a inverse expressed by $\sum_{i \in \mathbb{N}} \Phi^i$. We also have that $\mathrm{id} - \Phi = \partial \circ \mathcal{A}$. Then,

$$\partial \circ \left(\mathcal{A} \circ \sum_{i \in \mathbb{N}} \Phi^i \right) = (\partial \circ \mathcal{A}) \circ (\partial \circ \mathcal{A})^{-1} = \mathrm{id}$$

In particular, for all x, $\left(\mathcal{A} \circ \sum_{i \in \mathbb{N}} \Phi^i \right)(x)$ is a anti-derivative of x. $\square$

Chapter 4

Universality of the surreal numbers

What makes **No** a very interesting field is that fact that every ordered field can be embedded into it. Conway even call this field the field of “all numbers”. A particular set of “numbers” are the transseries. As well as surreal numbers, they are expressed in a formal way with a formal power series of possible ordinal length. They also have log-atomic numbers that turn out to correspond to the iterated logarithms and exponentials of ω. In particular, there is no log-atomic numbers that cannot be easily expressed in terms of ω. An other difference between transseries and surreal numbers is the use of the variable x instead of ω which make them much more actual functions, which they are not.
Following the literature, this chapter explains to what extend surreal numbers can be considered as universal and the connections there are between surreal numbers and transseries. More precisely,

- Section 4.1 is dedicated to some considerations about the meaning of the term “universal”.
- Section 4.2 introduce the transseries following van der Hoeven [49].
- Finally Section 4.3 makes the link between transseries and surreal numbers.

4.1 MacLane-like theorem

In the previous chapter, Theorems 3.3.23 and 3.3.24 ensure that $\mathbf{No} = \mathbb{R}\left(\left(\mathbf{No}\right)\right)$, or at least are isomorphic. This enables us to speak about formal series for surreal numbers. Formal series are already well studied and have good universality properties. In this section, we introduce MacLane’s theorem that is about formal power series and we give a universality theorem from Conway.

Definition 4.1.1. A field $\mathbb{K}$ is **universal** if every field which has same cardinal and characteristic[1] is isomorphic to some subfield of $\mathbb{K}$.

Remark 4.1.2. Note that we are no more interested in the characteristic of the ordered field $\mathbb{K}$ since every ordered field has characteristic 0.

MacLane proved two theorems about universality which are the following:

Theorem 4.1.3 ([36, MacLane, Theorem 2])**.** *A non-denumerable field $\mathbb{K}$ is universal if and only if it contains an algebraically closed subfield which has the same cardinal number as $\mathbb{K}$.*

Theorem 4.1.4 ([36, MacLane, Theorem 3])**.** *If the ordered Abelian divisible group G contains an element different from 0, while the coefficient field $\mathbb{K}$ is algebraically closed, then the Hahn field $\mathbb{K}\left(\left(G\right)\right)$ is universal.*

The previous theorem apply for algebraically closed field. For sure, **No** is not algebraically closed: It is real-closed. However, a universality property still holds in the world of ordered fields.

Definition 4.1.5. An ordered field $\mathbb{K}$ is **universal** if every ordered field which has same cardinal is isomorphic to some subfield of $\mathbb{F}$.

The field **No** is a proper class and there for has no cardinal. However we can still say that it is universal in the sense that every ordered field (for the Set Theory) are isomorphic to some subfield of **No**.

Theorem 4.1.6 ([18, Conway, Theorem 28])**.** *The field* **No** *is universal.*

[1]The characteristic of a field is the smallest integer n such that $n1 = 0$. If there is no such an integer, for instance in the case of $\mathbb{R}, \mathbb{C}$ or any surreal field we saw, its characteristic is set to 0.

4.2 Transseries

This section introduces the theory of transseries. We want to give a sufficient background to show how they relate to surreal numbers in the next section. Therefore, we do not pretend to be exhaustive and refer to [21, 41, 49] for a complete presentation and study. In this section we mostly follow the introduction of van der Hoeven [49].

4.2.1 Definition

Informally, transseries are formal objects made with $\mathbb{R}$ and a variable x and closed under field operations, exponential, logarithm and infinite (countable) sums (that is were comes from the name ‘transseries”). In fact, the field of transseries is an increasing union of Hahn series fields that is made to be stable under the operations we need. Therefore we have common notions between Hahn series and transseries.

Definition 4.2.1. The **formal well-ordered series** over a ordered multiplicative Abelian group $\mathcal{M}$ is given by

$$\mathbb{R}\left\{\{\mathcal{M}\}\right\} = \left\{ \sum_{\mathrm{m}\in\mathcal{M}} r_{\mathrm{m}}\mathrm{m} \;\middle|\; r_{\mathrm{m}} \in \mathbb{R} \quad \{\mathrm{m} \mid r_{\mathrm{m}} \neq 0\} \text{ is reverse-well-ordered} \right\}$$

An element $\mathrm{m} \in \mathcal{M}$ is called a **monomial** and $\mathcal{M}$ is called the **monomials group** of $\mathbb{R}\left\{\{\mathcal{M}\}\right\}$. Similarly to the case of surreal numbers, $f = \sum\limits_{\mathrm{m}\in\mathcal{M}} r_{\mathrm{m}}\mathrm{m}$ is

- **purely infinite** if for all $\mathrm{m} \leq 1$, $r_{\mathrm{m}} = 0$. We denote $\mathbb{R}\left\{\{\mathcal{M}\}\right\}_\infty$ the set of purely infinite numbers in $\mathbb{R}\left\{\{\mathcal{M}\}\right\}$. We also denote $\mathbb{R}\left\{\{\mathcal{M}\}\right\}_\infty^+$ the set of non-negative purely infinite series.
- **infinitesimal** if for all $\mathrm{m} \geq 1$, $r_{\mathrm{m}} = 0$.
- **appreciable** if for all $\mathrm{m} > 1$, $r_{\mathrm{m}} = 0$.

Note that for each $\mathcal{M}$, it is possible to find $\mathcal{M}'$ such that $\mathbb{R}\left\{\{\mathcal{M}\}\right\}$ and $\mathbb{R}\left((\mathcal{M}')\right)$ (see Definition 3.3.1) are isomorphic as ordered fields. For instance, $\mathbb{R}\left((\mathbb{R})\right)$ and $\mathbb{R}\left\{\left\{x^{\mathbb{R}}\right\}\right\}$ are isomorphic. The point of this definition is not to introduce a new object but a new way to describe it. More precisely, this expression enables us to handle directly the monomial instead of the exponents. We show the advantage of this point of view with the definition of transseries.

Definition 4.2.2 (Transseries)**.** Let $p \in \mathbb{N} \cup \{\omega\}$. Consider the following groups:

- $\mathcal{M}_0^p(x) = \left\{ \prod\limits_{k=0}^{n} \ln_k(x)^{a_k} \;\middle|\; n \in \mathbb{N} \quad n \leq p \quad a_k \in \mathbb{R} \quad a_n \neq 0 \right\}$
- For any ordinal α, $\mathcal{M}_{\alpha+1}^p(x) = \langle \mathcal{M}_\alpha^p(x), \{\exp(f) \mid f \in \mathbb{R}\left\{\{\mathcal{M}_\alpha^p(x)\}\right\}_\infty\}\rangle$ the group generated by $\mathcal{M}_\alpha^p(x)$ and the formal expressions $\exp(f)$ with the order define by

  $$\mathrm{m}\exp(f) < \mathrm{n}\exp(g) \quad \Leftrightarrow \quad f < g \vee (f = g \wedge \mathrm{m} < \mathrm{n}) \quad \Leftrightarrow \quad (f, \mathrm{m}) <_{lex} (g, \mathrm{n})$$

  for $\mathrm{mn} \in \mathcal{M}_\alpha^p(x)$ and $f, g \in \mathbb{R}\left\{\{\mathcal{M}_\alpha^p(x)\}\right\}_\infty$.
- For any limit ordinal α, $\mathcal{M}_\alpha^p(x) = \bigcup\limits_{\beta<\alpha} \mathcal{M}_\beta^p(x)$.

A **transseries** is an element of $\mathbb{R}\left\{\{\mathcal{M}_\alpha^p(x)\}\right\}$ for some ordinals α and p with $p \leq \omega$. It is said **alogarithmic** if it belong to some $\mathbb{R}\left\{\left\{\mathcal{M}_\alpha^0(x)\right\}\right\}$.

Definition 4.2.3 (Exponential and logarithmic depth)**.** Let f be a transseries. The minimum ordinal number α such that $f \in \mathbb{R}\left\{\{\mathcal{M}_\alpha^\omega(x)\}\right\}$ is called its **exponential depth**. The minimum ordinal number $p \leq \omega$ such that $f \in \mathbb{R}\left\{\{\mathcal{M}_\alpha^p(x)\}\right\}$ is called its **logarithmic depth**.

Remark 4.2.4. Note that we could have chosen any α greater or equal to the exponential depth of f to define its logarithmic depth.

Example 4.2.5.

- The transseries $\exp\left(\sum\limits_{n\in\mathbb{N}} \ln_n x\right)$ has exponential depth 1 and logarithmic depth ω.
- The transseries $\sum\limits_{n\in\mathbb{N}} \exp\left(-\exp_n(x)\right)$ is alogarithmic and has exponential depth ω.
- The transseries $\exp_k\left(\sum\limits_{n\in\mathbb{N}} \exp_n\left(\dfrac{1}{n+1}\ln_n x\right)\right)$ has logarithmic depth ω and exponential depth $\omega + k$.

- The transseries $\sum_{n\in\mathbb{N}} \exp\left(-\exp_n x + \ln_n x\right)$ has exponential and logarithmic depth ω.

Definition 4.2.6. As for surreal numbers we use the pre-orders $\prec, \preceq$ and the associated equivalence relation $\asymp$. We also use the equivalence relation $\sim$. There are defined similarly as in Definition 3.3.11.

There are several "natural" fields of transseries that we can consider. We give a list of them and by default will speak about field of the following kind when using the notion of transseries field.

Definition 4.2.7 (Transseries field)**.** A **transseries field** is a field of the following form:

- $\mathbb{R}^p_\alpha \llbracket x \rrbracket = \mathbb{R}\left\{\left\{\mathcal{M}^p_\alpha(x)\right\}\right\}$, the field transseries of exponential depth at most α and logarithmic depth at most p.
- $\mathbb{R}^p_{<\alpha} \llbracket x \rrbracket = \bigcup_{\beta<\alpha} \mathbb{R}^p_\beta \llbracket x \rrbracket$
- $\mathbb{R}^{\text{alog}}_\alpha \llbracket x \rrbracket = \mathbb{R}^0_\alpha \llbracket x \rrbracket$, the field of alogarithmic transseries of exponential depth at most α. Note that $\mathbb{R}^{\text{alog}}_\alpha \llbracket \ln_p x \rrbracket$ and $\mathbb{R}^p_\alpha \llbracket x \rrbracket$, for $p < \omega$, are naturally isomorphic.
- $\mathbb{R}^{\text{alog}}_{<\alpha} \llbracket x \rrbracket = \bigcup_{\beta<\alpha} \mathbb{R}^{\text{alog}}_\beta \llbracket x \rrbracket$
- $\mathbb{R}_\alpha \llbracket x \rrbracket = \bigcup_{p\in\mathbb{N}} \mathbb{R}^p_\alpha \llbracket x \rrbracket$, the field of transseries of finite logarithmic depth and exponential depth at most α.
- $\mathbb{R}_{<\alpha} \llbracket x \rrbracket = \bigcup_{\beta<\alpha} \mathbb{R}_\alpha \llbracket x \rrbracket$
- $\mathbb{R}^\omega \llbracket x \rrbracket = \bigcup_{n\in\mathbb{N}} \mathbb{R}^\omega_n \llbracket x \rrbracket$, the field of transseries of finite exponential depth.
- $\mathbb{R} \llbracket x \rrbracket = \mathbb{R}_{<\omega} \llbracket x \rrbracket = \bigcup_{n,p\in\mathbb{N}} \mathbb{R}^p_n \llbracket x \rrbracket$, the field of transseries of both finite exponential and logarithmic depth.

We defined many new fields, but actually, there is some redundancy. This fact is stated by Proposition 4.2.10. To prove it we need the following lemmas. A proof has already been give by van der Hoeven [49]. We provide a proof that only use the notations and notions we introduced.

Lemma 4.2.8. *Let $\mathfrak{m} \in \mathcal{M}^0_\alpha(x)$ and $\mathfrak{m}' \in \mathcal{M}^0_\beta(x)$ be two alogarithmic transmonomials with $\mathfrak{m} < \mathfrak{m}'$, α the exponential depth of $\mathfrak{m}$ and β the exponential depth of $\mathfrak{m}'$. Assume $\beta < \alpha < \omega$. Then there is some negative purely infinite transseries f and a real number s the that*

$$\mathfrak{m} = x^s \exp f$$

Proof. We proceed by induction on the ordered pair (α, β).

- If $\alpha = 1$ and $\beta = 0$. Then there are some real numbers r, s and $f \in \mathbb{R}\left\{\left\{\mathcal{M}^0_0(x)\right\}\right\}_\infty$ such that $\mathfrak{m} = x^s \exp f$ and $\mathfrak{m}' = x^r$. By definition of the order, we need $f \leq 0$. However, since the exponential depth of $\mathfrak{m}$ is exactly 1, f is non-zero. Hence, $f < 0$.
- Assume the property for all ordered pair $(\delta, \gamma) <_{lex} (\alpha, \beta)$ such that $\delta > \gamma$. Write

$$\mathfrak{m} = x^r \exp f \qquad \text{and} \qquad \mathfrak{m}' = x^s \exp g$$

  with $f \in \mathbb{R}^{\text{alog}}_{\alpha-1} \llbracket x \rrbracket$ and $g \in \mathbb{R}^{\text{alog}}_{\beta-1} \llbracket x \rrbracket \cup \{0\}$.

  - ➢ If $g < 0$, then since $f \leq g$ by assumption, we already have $f < 0$.
  - ➢ If $g = 0$ or if $\beta = 0$ (and then $g = 0$), the same argument as in the initialization step ensures $f < 0$.
  - ➢ If $g > 0$ (and then $\beta > 0$). Let $\mathfrak{m}_1$ be the dominant monomial of f. If $\mathfrak{m}_1$ has exponential depth less than $\alpha - 1$ then there is some monomial $\mathfrak{m}'_1 < \mathfrak{m}_1$ with exponential depth $\alpha - 1$ in the support of f. By induction hypothesis, $\mathfrak{m}'_1$ is infinitesimal. This contradicts the fact that f is purely infinite. Now take $\mathfrak{m}_2$ be the dominant monomial of g. If $f > 0$, then since $f < g$, we must have $\mathfrak{m}_1 < \mathfrak{m}_2$ (the equality cannot hold since they do not have the same exponential depth). Then again, by induction hypothesis, $f < 0$ what is impossible. Then $f \leq 0$. Equality cannot hold since $\mathfrak{m}$ has positive exponential depth. Therefore $f < 0$.

□

Lemma 4.2.9 ([49, van der Hoeven, Proposition 2.1])**.** *Any alogarithmic transseries has exponential depth at most ω.*

Proof. Assume that the lemma is false. Let α minimum that there is some $f \in \mathbb{R}_\alpha^{\mathrm{alog}} [\![x]\!]$ with exponential depth greater than ω. In particular $\alpha > \omega$.
By assumption on f there is $\mathfrak{m} \in \operatorname{supp} f$ such that $\mathfrak{m}$ has exponential depth greater than ω. Then, there are an ordinal $\beta < \alpha$, a monomial $\mathfrak{m}' \in \mathcal{M}_0^0(x)$ and series $g \in \mathbb{R}\left\{\left\{\mathcal{M}_\beta^0(x)\right\}\right\}_\infty$ such that

$$\mathfrak{m} = \mathfrak{m}' \exp(g)$$

By minimality of α, g cannot have exponential depth greater than ω: $g \in \mathbb{R}_\omega^{\mathrm{alog}} [\![x]\!]$. Then it has exponential depth exactly ω (since it must be infinite). Write

$$g = \sum_{i<\nu} r_i \mathfrak{m}_i$$

with $(\mathfrak{m}_i)_{i<\nu}$ decreasing and ν an ordinal. Since g has exponential depth ω and all the transmonomials $\mathfrak{m}_i$ have finite exponential depth, we need $\nu \geq \omega$. Moreover, since g has exponential depth ω, there is some increasing function $\varphi : \mathbb{N} \to \nu$ such that the sequence of the exponential depths of the transmonomials $\mathfrak{m}_{\varphi(n)}$ is increasing with $n \in \mathbb{N}$. Using Lemma 4.2.8, for all natural number n, there is some negative transseries h_n and a real number s_n such that

$$\mathfrak{m}_{\varphi(n)} = x^{s_n} \exp h_n$$

In particular, $\mathfrak{m}_{\varphi(n)}$ is infinitesimal. But this contradicts the fact that g is purely infinite. □

Proposition 4.2.10 ([49, van der Hoeven, Proposition 2.1])**.** *Any transseries with finite logarithmic depth has exponential depth at most ω. In particular, for $p \in \mathbb{N}$ and $\alpha > \omega$, $\mathbb{R}_\alpha^p [\![x]\!] = \mathbb{R}_{<\alpha}^p [\![x]\!] = \mathbb{R}_\omega^p [\![x]\!]$ and $\mathbb{R}_\alpha [\![x]\!] = \mathbb{R}_{<\alpha} [\![x]\!] = \mathbb{R}_\omega [\![x]\!]$.*

Proof. Let $f \in \mathbb{R}_\alpha^p [\![x]\!]$ for some $p \in \mathbb{N}$. We take α to be minimal. We notice that there is a natural embedding $\mathcal{M}_0^p(x) \subseteq \mathcal{M}_p^0(\ln_p x)$. Therefore, by induction, we have $\mathcal{M}_\alpha^p(x) \subseteq \mathcal{M}_{p\oplus\alpha}^0(\ln_p x)$. Therefore $\mathbb{R}_\alpha^p [\![x]\!] \subseteq \mathbb{R}_{p\oplus\alpha}^{\mathrm{alog}} [\![\ln_p x]\!]$. Using Lemma 4.2.9 and the minimality of α, we get that $p \oplus \alpha \leq \omega$, hence $\alpha \leq \omega$. □

On the contrary, is the logarithmic depth is not bounded by any natural number, the fields are different.

Proposition 4.2.11 ([49, van der Hoeven, Proposition 2.2])**.** *If $\alpha < \beta$ then $\mathbb{R}_\alpha^\omega [\![x]\!] \subsetneq \mathbb{R}_\beta^\omega [\![x]\!]$.*

To show that, we give an example of function that makes the distinction.

Definition 4.2.12 (Composition by a logarithmic function)**.** For any $f \in \mathbb{R}_\alpha^\omega [\![x]\!]$, we let $f \circ \ln_p x \in \mathbb{R}_\alpha^\omega [\![\ln_p x]\!]$ be the image of f by the natural isomorphism of $\mathbb{R}_\alpha^\omega [\![x]\!]$ into $\mathbb{R}_\alpha^\omega [\![\ln_p x]\!]$. We see $f \circ \ln_p$ as an element of $\mathbb{R}_\alpha^\omega [\![x]\!]$.

Definition 4.2.13. Consider the following family of functions:

- Let $f_0 = x$.
- For any ordinal α such that f_β has been defined for $\beta < \alpha$, we define f_α as follows :

$$f_\alpha = x + \sum_{\omega\otimes\beta+n<\alpha} \exp\left(\frac{1}{n+1} \frac{f_{\omega\otimes\beta+n} \circ \ln x}{\exp\left(f_{\omega\otimes\beta} \circ \ln_3 x\right)}\right)$$

Note that this is consistent with the definition of f_0 and that by definition, the sequence $(f_\alpha)_\alpha$ is increasing.

Lemma 4.2.14. *The family of function defined in the previous definition is well defined.*

Proof. We need to show that for all α such that f_γ has been defined for $\gamma < \alpha$ and for all β, β', n, n' such that $\alpha > \omega \otimes \beta + n > \omega \otimes \beta' + n'$, we have

$$\frac{f_{\omega\otimes\beta+n} \circ \ln x}{(n+1)\exp\left(f_{\omega\otimes\beta} \circ \ln_3 x\right)} < \frac{f_{\omega\otimes\beta'+n'} \circ \ln x}{(n'+1)\exp\left(f_{\omega\otimes\beta'} \circ \ln_3 x\right)} \qquad (*)$$

and that these quantities are purely infinite.

- The property is trivially true for $\alpha = 0$.
- Assume the property for all $\gamma < \alpha$. We split into two cases:
  - ➢ If $\alpha = \omega \otimes \beta$, *i.e* α is a limit ordinal, then the induction hypothesis immediately ensures that f_α is well defined and that the inequality $(*)$ holds.

➢ If $\alpha = \omega \otimes \beta + n + 1$, then

$$f_\alpha = f_{\omega\otimes\beta+n} + \exp\left(\frac{1}{n+1}\frac{f_{\omega\otimes\beta+n}\circ \ln x}{\exp\left(f_{\omega\otimes\beta}\circ\ln_3 x\right)}\right)$$

In particular, it is well defined. By definition,

$$\frac{f_{\omega\otimes\beta+n}\circ \ln x}{(n+1)\exp\left(f_{\omega\otimes\beta}\circ\ln_3 x\right)} = \frac{\ln x}{(n+1)\exp\left(f_{\omega\otimes\beta}\circ\ln_3 x\right)} + \sum_{\omega\otimes\beta'+n'<\omega\otimes\beta+n}\frac{\exp\left(\frac{1}{n'+1}\frac{f_{\omega\otimes\beta'+n'}\circ \ln x}{\exp\left(f_{\omega\otimes\beta'}\circ\ln_3 x\right)}\right)}{(n+1)\exp\left(f_{\omega\otimes\beta}\circ\ln_3 x\right)}$$

For all β', since $f_{\omega\otimes\beta'}\circ\ln_3 x \sim \ln_3 x$ and all the term of the series are positive,

$$\ln_2 x \leq \exp\left(f_{\omega\otimes\beta'}\circ\ln_3 x\right) \prec (\ln_2 x)^2$$

and $$\exp\left(\frac{\ln x}{(n'+1)(\ln_2 x)^2}\right) \prec \exp\left(\frac{1}{n'+1}\frac{f_{\omega\otimes\beta'+n'}\circ \ln x}{\exp\left(f_{\omega\otimes\beta'}\circ\ln_3 x\right)}\right) \leq \exp\left(\frac{\ln x}{(n'+1)\ln_2 x}\right)$$

Hence, $\dfrac{\exp\left(\frac{1}{n'+1}\frac{f_{\omega\otimes\beta'+n'}\circ \ln x}{\exp\left(f_{\omega\otimes\beta'}\circ\ln_3 x\right)}\right)}{(n+1)\exp\left(f_{\omega\otimes\beta}\circ\ln_3 x\right)}$ is a purely infinite positive transmonomial and finally, $\dfrac{f_{\omega\otimes\beta+n}\circ \ln x}{(n+1)\exp\left(f_{\omega\otimes\beta}\circ\ln_3 x\right)}$ is purely infinite. It now remains to show that the inequality $(*)$ holds. Let $\omega\otimes\beta'+n' < \omega\otimes\beta+n$. Note that this is the only case not covered by the induction hypothesis.

∵ If $\beta' = \beta$, then $n' < n$ and

$$\frac{f_{\omega\otimes\beta+n}\circ \ln x}{(n+1)\exp\left(f_{\omega\otimes\beta}\circ\ln_3 x\right)} \sim \frac{\ln x}{(n+1)\exp\left(f_{\omega\otimes\beta}\circ\ln_3 x\right)} = \frac{\ln x}{(n+1)\exp\left(f_{\omega\otimes\beta'}\circ\ln_3 x\right)} < \frac{\ln x}{(n'+1)\exp\left(f_{\omega\otimes\beta'}\circ\ln_3 x\right)} \sim \frac{f_{\omega\otimes\beta'+n'}\circ \ln x}{(n'+1)\exp\left(f_{\omega\otimes\beta'}\circ\ln_3 x\right)}$$

By definition of the order,

$$\frac{f_{\omega\otimes\beta+n}\circ \ln x}{(n+1)\exp\left(f_{\omega\otimes\beta}\circ\ln_3 x\right)} < \frac{f_{\omega\otimes\beta'+n'}\circ \ln x}{(n'+1)\exp\left(f_{\omega\otimes\beta'}\circ\ln_3 x\right)}$$

∵ If $\beta' < \beta$, then $f_{\omega\otimes\beta}\circ\ln_3 x > f_{\omega\otimes\beta'}\circ\ln_3 x$ and then

$$\frac{f_{\omega\otimes\beta+n}\circ \ln x}{(n+1)\exp\left(f_{\omega\otimes\beta}\circ\ln_3 x\right)} \asymp \frac{\ln x}{\exp\left(f_{\omega\otimes\beta}\circ\ln_3 x\right)} \prec \frac{\ln x}{\exp\left(f_{\omega\otimes\beta'}\circ\ln_3 x\right)} \asymp \frac{f_{\omega\otimes\beta'+n'}\circ \ln x}{(n'+1)\exp\left(f_{\omega\otimes\beta'}\circ\ln_3 x\right)}$$

Since we are comparing positive transseries,

$$\frac{f_{\omega\otimes\beta+n}\circ \ln x}{(n+1)\exp\left(f_{\omega\otimes\beta}\circ\ln_3 x\right)} < \frac{f_{\omega\otimes\beta'+n'}\circ \ln x}{(n'+1)\exp\left(f_{\omega\otimes\beta'}\circ\ln_3 x\right)}$$

□

Example 4.2.15. For instance, we have

$$\begin{aligned}
f_1 &= x + \exp\left(\frac{\ln_x}{\ln_2 x}\right)\\
f_2 &= x + \exp\left(\frac{\ln x}{\ln_2 x}\right) + \exp\left(\frac{1}{2}\frac{\ln x + \exp\left(\frac{\ln_2 x}{\ln_3 x}\right)}{\ln_2 x}\right)\\
f_\omega &= f_2 + \cdots\\
f_{\omega+1} &= f_\omega + \exp\left(\frac{\ln x + \exp\left(\frac{\ln_2 x}{\ln_3 x}\right) + \cdots}{\exp\left(\ln_3 x + \exp\left(\frac{\ln_4 x}{\ln_5 x}\right) + \cdots\right)}\right)\\
&= f_\omega + \exp\left(\frac{\ln x + \exp\left(\frac{\ln_2 x}{\ln_3 x}\right) + \cdots}{\ln_2(x)\exp\left(\exp\left(\frac{\ln_4 x}{\ln_5 x}\right) + \cdots\right)}\right)
\end{aligned}$$

Proof of Proposition 4.2.11. We make use of the family of function we have just defined. We claim the following :

(i) If α is a limit ordinal (including 0), then f_α has exponential depth α.

(ii) If α is a successor ordinal, then f_α has exponential depth $\alpha + 1$.

We prove this property by induction on α.

- $f_0 = x$ has exponential depth 0
- Assume the property for all $\gamma < \alpha$. We split into two cases:
  - ➢ If $\alpha = \omega \otimes \beta$ is a limit ordinal with $\beta \geq 1$. By induction hypothesis, the exponential depth of f_α is
  $$\sup \{\omega \otimes \beta' + n + 1 \mid \beta' < \beta, n \in \mathbb{N}\} = \omega \otimes \beta$$
  - ➢ If $\alpha = \omega \otimes \beta + 1$ with $\beta \geq 0$. Then
  $$f_\alpha = f_{\omega\otimes\beta} + \exp\left(\frac{f_{\omega\otimes\beta} \circ \ln x}{n+1} \exp\left(-f_{\omega\otimes\beta} \circ \ln_3 x\right)\right)$$
  and by induction hypothesis, f_α has exponential depth
  $$\max\left(\omega \otimes \beta, \max\left(\omega \otimes \beta, \omega \otimes \beta + 1\right) + 1\right) = \omega \otimes \beta + 2 = \alpha + 1$$
  - ➢ If $\alpha = \omega \otimes \beta + n + 1$ with $n \geq 1$ and $\beta \geq 0$. Then
  $$f_\alpha = f_{\omega\otimes\beta+n} + \exp\left(\frac{f_{\omega\otimes\beta+n} \circ \ln x}{n+1} \exp\left(-f_{\omega\otimes\beta} \circ \ln_3 x\right)\right)$$
  and by induction hypothesis, f_α has exponential depth
  $$\max\left(\omega \otimes \beta + n + 1, \max\left(\omega \otimes \beta + n + 1, \omega \otimes \beta + 1\right) + 1\right) = \omega \otimes \beta + 2 = \alpha + 1$$

Therefore:

- If α is a limit ordinal, for any ordinal number $\beta < \alpha$, $f_\alpha \in \mathbb{R}^\omega_\alpha [\![x]\!] \setminus \mathbb{R}^\omega_\beta [\![x]\!]$. In particular $\mathbb{R}^\omega_\beta [\![x]\!] \subsetneq \mathbb{R}^\omega_\alpha [\![x]\!]$.
- If $\alpha = \gamma + 1$ is a successor ordinal, for any $\beta < \alpha$, $f_\alpha \in \mathbb{R}^\omega_{\alpha+1} [\![x]\!] \setminus \mathbb{R}^\omega_\beta [\![x]\!]$. In particular $\mathbb{R}^\omega_\beta [\![x]\!] \subsetneq \mathbb{R}^\omega_{\alpha+1} [\![x]\!]$.
- Finally, for all limit ordinal α $\exp(f_\alpha)$ has exponential depth $\alpha + 1$. hence, for any ordinal number $\beta < \alpha$, $\mathbb{R}^\omega_\beta [\![x]\!] \subsetneq \mathbb{R}^\omega_\alpha [\![x]\!]$.

Using these three above items, for any ordinal numbers $\beta < \alpha$, we indeed have $\mathbb{R}^\omega_\beta [\![x]\!] \subsetneq \mathbb{R}^\omega_\alpha [\![x]\!]$. □

Remark 4.2.16. In his PhD thesis, van der Hoeven gave an other example of functions that makes the distinction:

- $f_0 = 0$
- $f_\alpha = x^{1/2} - \sum\limits_{\beta<\alpha} \exp\left(f_\alpha \circ \ln x\right)$

4.2.2 Tree representation

Similarly to surreal numbers (Section 3.8.2), transseries have a natural well-ordered tree representation. The log-atomic number, in the context of transseries, are the iterated exponential and logarithm. Except this little change, the definition of the tree representation for transseries is almost unchanged compare to Definition 3.8.9.

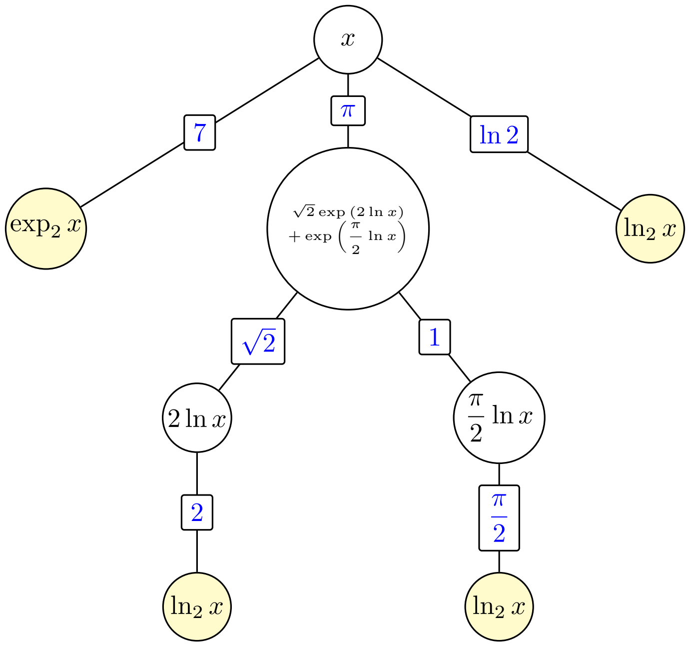
Definition 4.2.17 (Well-ordered tree representation of transseries)**.** Let f be a transseries. The well-ordered tree representation of f given by the following:

- f is at the root
- If u is not an iterated exponential nor logarithm, and is a node, and if v is purely infinite such that $r \exp v$ is a term of u for some $r_i n \mathbb{R}^*$, then (u, v) is an edge labeled by r.
- If $u = \exp_n x$ for some $n \in \mathbb{Z}$, then it has to be a leaf.
- If u is 0, it must be either the root and the only node, either a child of the root.

Example 4.2.18. The analogous to Example 3.8.10 is the following. For the transseries

$$f = 7\exp_2 x + \pi \exp\left(\sqrt{2}\exp(2\ln x) + \exp\left(\frac{\pi}{2}\ln x\right)\right) + (\ln 2)\ln x$$

the tree representation is:



4.2.3 Operations over transseries

Because of the Hahn series fields underlying in all the transseries fields, all of them are indeed fields. Therefore all the field operations are available on transseries fields. There are other operations that we can make over transseries. This section is about some of them.

Exponentiation Exponentiation is defined the expected way: Let $f = \sum_{\mathfrak{m}\in\mathcal{M}} r_{\mathfrak{m}}\mathfrak{m}$ be a transseries, $\mathcal{M}$ being some $\mathcal{M}^p_\alpha(x)$. We write $f = f_\infty + f_a$ with $f_a = \sum_{\mathfrak{m}\preceq 1} r_{\mathfrak{m}}\mathfrak{m}$ the appreciable part and $f_\infty = \sum_{\mathfrak{m}\succ 1} r_{\mathfrak{m}}\mathfrak{m}$ the purely infinite part. $\exp f_a$ can be defined formally by the series of exp:

$$\exp f_a = \sum_{n\in\mathbb{N}} \frac{1}{n!} f_a^n \in \mathbb{R}\{\{\mathcal{M}\}\}$$

Now, if $f_\infty \in \mathbb{R}\{\{\mathcal{M}^p_\alpha(x)\}\}$, then by definition, $\exp f_\infty \in \mathcal{M}^p_{\alpha+1}(x)$. Finally we of course define:

$$\exp f = (\exp f_\infty)(\exp f_a)$$

Logarithm The natural logarithm can be extended to the positive transseries.

Definition 4.2.19 (Logarithm of a monomial)**.** Let $\mathfrak{m} \in \mathcal{M}^p_\alpha(x)$ be positive. We define $\ln \mathfrak{m}$ by induction on α.

- If $\alpha = 0$ then $\mathfrak{m} = \prod_{k=0}^{n} (\ln_k x)^{a_k}$ for some $n \in \mathbb{N}$, $a_k \in \mathbb{R}$ and $a_n \neq 0$. Then we define

$$\ln \mathfrak{m} = \sum_{k=0}^{n} a_k \ln_{k+1} x$$

- If $\alpha = \beta + 1$, then $\mathfrak{m} = \left(\prod_{k=0}^{n} (\ln_k x)^{a_k}\right) \exp f$ for some $n \in \mathbb{N}$, $a_k \in \mathbb{R}$, $a_n \neq 0$ and $f \in \mathbb{R}^p_\beta[\![x]\!]_\infty$. We then define

$$\ln \mathfrak{m} = f + \sum_{k=0}^{n} a_k \ln_{k+1} x$$

- If α is a limit ordinal, then there is an ordinal $\beta < \alpha$ such that $\mathrm{m} \in \mathcal{M}^p_\beta(x)$ and then $\ln \mathrm{m}$ has already been defined.

Definition 4.2.20. Let f be a positive transseries and write $f = r_0\mathrm{m}_0(1+\varepsilon)$ with $r_0\mathrm{m}_0$ the first term of f and ε an infinitesimal. Then

$$\ln f = \ln \mathrm{m} + \ln r_0 + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{\varepsilon^k}{k}$$

As for surreal number, this is well defined and is built to be the compositional inverse if $\exp$.

Differentiation We can define the differentiation formally over transseries.

Definition 4.2.21. Let f be a transseries. We define the derivative f' of f inductively as follows:

- $0' = 0$
- $\ln_k' x = \dfrac{1}{\prod\limits_{i=0}^{k-1} \ln_i x}$
- $\left(\sum\limits_f r_f \exp f\right)' = \sum\limits_f r_f f' \exp f$

The fact that it is well defined is not trivial and is based of the idea of path that has already been done for surreal numbers. Since this idea is the same, we do not give the details. For a complete introduction we again refer to [49, Section 2.4].

Infinite sums The notion of sum of series of transseries is similar to the notion of summable family for surreal numbers (see Definition 3.9.1)

Definition 4.2.22 (Summable family or transseries)**.** Let $\{f_i\}_{i\in I}$ be a family of transseries. For $i \in I$ write

$$x_i = \sum_{\mathrm{m}\in\mathcal{M}} r_{i,\mathrm{m}}\mathrm{m}$$

where

$$\mathcal{M} = \bigcup_{p\in\mathbb{N}}\bigcup_{\alpha\in\mathbf{Ord}} \mathcal{M}^p_\alpha(x)$$

The family $\{f_i\}_{i\in I}$ is **summable** *iff*

(i) $\bigcup\limits_{i\in I} \operatorname{supp} f_i$ is a reverse well ordered set.

(ii) For all $\mathrm{m} \in \bigcup\limits_{i\in I} \operatorname{supp} f_i$, $\{\, i \in I \mid \mathrm{m} \in \operatorname{supp} f_i\}$ is a finite set.

In this case, its sum is defined as $\sum\limits_{i\in I} f_i = \sum\limits_{\mathrm{m}\in\mathcal{M}} s_\mathrm{m}\mathrm{m}$ where for all $\mathrm{m} \in \mathcal{M}$,

$$s_\mathrm{m} = \sum_{i\in I \ \mid\ \mathrm{m}\in\operatorname{supp} f_i} r_{i,\mathrm{m}}$$

which is a finite sum.

Composition It is possible to define the composition of transseries. It is actually impossible to define it in the general case but if f is a transseries in a field of transseries $\mathbb{T}$ and $g \in \mathbb{T}$ is a positive infinite transseries, then we can define $f \circ g$. The intuition is that is that if f represents some behavior of some dynamical system when "x goes to infinity", then $f \circ g$ is the behavior of some dynamical system when we change how time goes, slowing it or accelerating it, but still going to infinity.

Definition 4.2.23. Let $f, g \in \mathbb{T}$ with g positive infinite. $f \circ g$ is defined by transfinite induction by:

- $0 \circ g = 0$
- $(\ln_k x) \circ g = \ln_k g$
- $\left(\sum\limits_f r_f \exp f\right) \circ g = \sum\limits_f r_f \exp(f \circ g)$

The fact that this is well defined is not trivial and uses a combinatorial approach. We refer to [49, Section 2.5.1] for the details. Composition over transseries have a lot of expected properties. They are summed up in the following proposition:

Proposition 4.2.24. *Let $f \in \mathbb{R}^p_\alpha [\![x]\!]$ and $(g,h) \in \mathbb{R}^{p'}_{\alpha'} [\![x]\!] \times \mathbb{R}^{p''}_{\alpha''} [\![x]\!]$ be positive infinite.*

(i) For all families $(f_i)_{i\in I}$ of $\mathbb{R}^p_\alpha [\![x]\!]$, if $\sum_{i\in I} f_i$ makes sense (i.e $\{f_i\}_{i\in I}$ is summable) then

$$\left(\sum_{i\in I} f_i\right) \circ g = \sum_{i\in I} f_i \circ g$$

(ii) $f \circ (g \circ h) = (f \circ g) \circ h$

(iii) $(f \circ g)' = g' \times f' \circ g$

(iv) If $\varepsilon \prec \dfrac{m'}{m}$ for all monomial $m \in \operatorname{supp} f$ and $\varepsilon \prec 1$, then

$$f \circ (x+\varepsilon) = \sum_{k\in\mathbb{N}} \frac{f^{(k)}}{k!}\varepsilon^k$$

(v) $f \circ g \in \mathbb{R}^{\min(p+p',\omega)}_{\min(\alpha\oplus\alpha',\max(\alpha,\alpha')\oplus\omega)} [\![x]\!]$

4.3 Transseries and surreal numbers

Pursuant to Theorem 4.1.6, since the transseries fields are ordered field, they can be embedded in **No**. But of course this embedding is quite trivial and can be defined by transfinite induction:

- 0 is represented by itself.
- $\ln_k x$ is represented by $\ln_k \omega$.
- $\sum_{i<\nu} r_i \exp f_i$ is represented by $\sum_{i<\nu} r_i \exp x_i$ where x_i is the embedding of f_i.

In some sense, it is possible to make the other direction: surreal numbers are transseries. More precisely, it is possible to axiomatize what we call a transseries and then check that surreal numbers are indeed a field of transseries in that sense.

Definition 4.3.1 ([41, Schmeling, Definition 2.2.1]). Let $\mathcal{M}$ be some monomial group, then $\mathbb{R}\{\{\mathcal{M}\}\}$ is called a **Schmeling transseries field** if it satisfies the following axioms:

T1. $\operatorname{dom}\ln = (\mathbb{R}\{\{\mathcal{M}\}\})^*_+$

T2. For any $\mathrm{m} \in \mathcal{M}$, $\ln \mathrm{m}$ is purely infinite: $\ln \mathcal{M} \subseteq \mathbb{R}\{\{\mathcal{M}\}\}_\infty$

T3. For any $f \in \mathbb{R}\{\{\mathcal{M}\}\}$ such that $f \prec 1$, $\ln(1+f) = \sum_{k=1}^{+\infty} \dfrac{f^k}{k}$

T4. For any sequence $(\mathrm{m}_i)_{i\in\mathbb{N}}$ of elements of $\mathcal{M}$ such that for any i, $\mathrm{m}_{i+1} \in \operatorname{supp}\ln \mathrm{m}_i$, there for natural number i_0 such that

$$\forall i \geq i_0 \quad \exists \gamma_i \in \mathbb{R}\{\{\mathcal{M}\}\} \qquad (\operatorname{supp}\gamma_i \succ \mathrm{m}_{i+1}) \wedge (\mathrm{m}_i = \exp(\gamma_i \pm \mathrm{m}_{i+1}))$$

Remark 4.3.2. $\mathbb{R}$ can be replaced by any totally ordered field $\mathbb{K}$ stable under a function $\exp$ and $\ln$ with $\operatorname{dom}\exp = \mathbb{K}$ and $\operatorname{dom}\ln = \mathbb{K}^*_+$.

There is an alternative notion of exp-log series which follows the following axiomatization:

Definition 4.3.3 ([33, Kuhlmann and Matusinski, Definition 5.1]). Let $\mathcal{M}$ be some monomial group, then $\mathbb{R}\{\{\mathcal{M}\}\}$ is called a (generalized) **exp-log series field** if it satisfies Axioms **T1.**, **T2.** and **T3.** and the variant of Axiom **T4.**:

ELT4. For any sequence $(\mathrm{m}_i)_{i\in\mathbb{N}}$ of elements of $\mathcal{M}$ such that for any i, $\mathrm{m}_{i+1} \in \operatorname{supp}\ln \mathrm{m}_i$, there for natural number i_0 such that

$$\forall i \geq i_0 \qquad \ln \mathrm{m}_{i+1} = \mathrm{m}_i$$

By construction, **No** does satisfy Axioms **T1.**, **T2.** and **T3.**. Kuhlmann and Matusinski conjectured that **No** was in fact an exp-log series field, *i.e* that **No** would satisfy Axiom **ELT4.**, [33, Conjecture 5.2]. It turned out that is was not true but **No** still satisfies **T4.**. This has been proved by Berarducci and Mantova.

Proposition 4.3.4 ([12, Berarducci and Mantova, Propositions 8.2 and 8.6]). *Let* $\mathbb{R}\left\langle\langle\mathbb{L}\rangle\right\rangle$ *be the smallest containing* $\mathbb{R}(\mathbb{L})$ *and stable under* $\exp, \ln$ *and infinite sums.* $\mathbb{R}\left\langle\langle\mathbb{L}\rangle\right\rangle$ *satisfies* ***ELT4.***. *More precisely, for any surreal* x, $x \in \mathbb{R}\left\langle\langle\mathbb{L}\rangle\right\rangle$ *iff for any path* $P \in \mathcal{P}(x)$*, there is some* $i \in \mathbb{N}$ *such that* $P(i) \in \mathbb{L}$*. Moreover,* $\mathbb{R}\left\langle\langle\mathbb{L}\rangle\right\rangle$ *is maximal for this property.*

Proposition 4.3.5 ([12, Berarducci and Mantova, Theorem 8.7]). **No** *does not satisfy* ***ELT4.****, more precisely,*

$$\mathbb{R}\left\langle\langle\mathbb{L}\rangle\right\rangle \subsetneq \mathbf{No}$$

Theorem 4.3.6 ([12, Berarducci and Mantova, Theorem 8.10]). **No** *is a Schmeling transseries field.*

With the previous theorem we get what we announced: transseries are particular example of surreal numbers and surreal numbers are a Schmeling transseries field. The correspondence is not perfect but **No** still satisfies basic properties we would expect from a transseries field. The main difference resides in the fact that there are special log-atomic numbers that does not correspond to any finite iteration of any hyper-exponential or any hyper-logarithm, such as $\lambda_{\frac{1}{2}}$ (see Definition 3.7.14).

Chapter 5

Substructures stable under advanced operations

This chapter contains our main contributions about surreal numbers. As announced, we provide fields of surreal numbers that are stable under $\exp, \ln, \partial$ and the anti-derivative (as defined in Corollary 3.10.14). To be more precise, given an ε-number λ and and a family of Abelian subgroups of **No**, $(\Gamma_i)_{i\in I}$, we provide a necessary and sufficient condition so that $\bigcup_{i\in I}\mathbb{R}_\lambda^{\Gamma_i}$ to be stable under $\exp$ and $\ln$. We focus on fields of the form $\mathbb{R}_\lambda^{\Gamma_i}$ because they are very easy to handle: We just have to list the exponents and coefficients and there is no doubt whether or not an element belongs the field. With this condition and some fastidious work on the derivative, ∂, we end up with a sufficient condition for a field to be stable under derivative and anti-derivative. We see these fields that have these four stability properties as our privileged fields to work on asymptotic behaviors of continuous models of computations in some future work.

- Section 5.1 studies the stability of the fields of surreal numbers we have seen so fare under exponential and logarithm. It also provides a decomposition of $\mathbf{No}_\lambda$ into an increasing hierarchy of subfields, all of them closed under exponential and logarithm.
- Section 5.2 studies the length of the series of the anti-derivatives in various cases. This will be helpful in the next section.
- Section 5.3 states and provides the proof of our main theorem. It establishes a sufficient condition under which a field of surreal numbers is stable under $\exp$, $\ln$, ∂ and the anti-derivative.

The main results of this chapter are:

- Theorem 5.1.6 and Proposition 5.1.7 which give a necessary and sufficient condition under which a special kind of union of surreal fields is stable under $\exp$ and $\ln$ and apply it to an already defined field to prove its stability.
- Theorems 5.1.10 and 5.1.11 prove that the fields considered by Theorem 5.1.6 can provide a decomposition of $\mathbf{No}_\lambda$ into an increasing hierarchy of subfields, all of them closed under exponential and logarithm. This hierarchy is strict what ensures its interest.
- Theorem 5.3.1 which provides a special kind of fields of surreal numbers which are guaranteed to be stable under $\exp$, $\ln$, ∂, and the anti-derivative. Example 5.3.3 provides the announced field stable under all these operations and that only include ordinal up to ε_ω.

5.1 Structure stable under exponential and logarithm

5.1.1 Stability of $\mathbf{No}_\lambda$ by exponential and logarithm

We first recall some results by van den Dries and Ehrlich.

Lemma 5.1.1 ([48, van den Dries and Ehrlich, Lemmas 5.2, 5.3 and 5.4])**.** *For all surreal number $a \in$ **No**,*

- $|\exp a|_{+-} \leq \omega^{\omega^{2|a|_{+-}\oplus 3}}$
- $|\ln \omega^a|_{+-} \leq \omega^{4\omega|a|_{+-}|a|_{+-}}$
- $|\ln a|_{+-} \leq \omega^{\omega^{3|a|_{+-}\oplus 3}}$

Corollary 5.1.2 ([48, van den Dries and Ehrlich, Corollary 5.5]). *For λ an ε-number, $\mathbf{No}_\lambda$ is stable under* $\exp$ *and* $\ln$.

Theorem 5.1.3. *The following are equivalent:*

- $\mathbf{No}_\lambda$ *is stable by* $\exp$*, and* $\ln$
- $\mathbf{No}_\lambda$ *is a subfield of* $\mathbf{No}$
- λ *is some ε-number.*

Proof. Using Theorem 3.4.1 we already know that $\mathbf{No}_\lambda$ is a field if an only if λ is an ε-number. Corollary 5.1.2 ensures that if λ is an ε-number, $\mathbf{No}_\lambda$ is stable under exponential and logarithm. The last thing to prove is that if λ is not an ε-number, then $\mathbf{No}_\lambda$ is not stable under one of theses functions. Then, let λ be an ordinal which is not an ε-number. Write it in the Cantor normal form as

$$\lambda = \sum_{i=0}^{n} \omega^{\alpha_i} n_i$$

with n a natural number as well as the coefficients n_i and $(\alpha_0, \dots, \alpha_n)$ being a finite decreasing sequence of ordinals. Since λ is not an ε-number, $\alpha_0 < \lambda$. In particular, $\alpha_0 \in \mathbf{No}_\lambda$. If $\lambda = \omega^{\alpha_0}$ then Lemma 5.1.1 give that $\exp(\alpha_0)$ has length at least λ. Therefore $\mathbf{No}_\lambda$ is not stable under exponential. Otherwise, $\lambda > \omega^{\alpha_0}$ and then $\exp(\omega^{\alpha_0}) = \omega^{\omega^{g(\alpha_0)}}$. By Proposition 3.6.6, $g(\alpha_0)$ is an ordinal number and

$$g(\alpha_0) = \begin{cases} \alpha_0 + 1 & \text{if } \lambda' \leq \alpha_0 < \lambda' + \omega \text{ for some } \varepsilon\text{-number } \lambda' \\ \alpha_0 & \text{otherwise} \end{cases}$$

In both cases we have $\alpha_0 < \omega^{g(\alpha_0)}$. Therefore, by Lemma 5.1.1, $\exp(\omega^{\alpha_0})$ has length at least $\omega^{\omega^{\alpha_0}}$ which is greater than λ. □

5.1.2 An instability result of the decomposition of $\mathbf{No}_\lambda$

One point about Theorem 3.4.6 is that it establishes that $\mathbf{No}_\lambda$ can be expressed as an increasing union of fields. However, even if $\mathbf{No}_\lambda$ is stable under exp and ln (Theorem 5.1.3) none of the fields in this union has stability properties beyond the fact that they are fields. Indeed, we have the following proposition:

Proposition 5.1.4. $\mathbb{R}_\lambda^{\mathbf{No}_\mu}$ *is never closed under* $\exp$ *for $\mu < \lambda$ a multiplicative ordinal.*

Proof. If μ is a multiplicative ordinal but not an ε-number, $\mu = \omega^{\omega^\alpha}$ for some ordinal $\alpha < \mu$. Since $g(\omega^\alpha) \geq \omega^\alpha$ (Proposition 3.6.6), we have $\omega^{g(\alpha)} \geq \omega^{\omega^\alpha} = \mu$. In particular, $\omega^{g(\alpha)} \notin \mathbf{No}_\mu$. Moreover, Proposition 3.6.5 ensures that $\exp(\omega^\alpha) = \omega^{\omega^{g(\alpha)}}$. Therefore, $\exp(\omega^\alpha) \notin \mathbb{R}_\lambda^{\mathbf{No}_\mu}$.
Now, if μ is an ε-number. Take $x = \sum_{0<i<\mu} \omega^{\omega^{-i}}$. Then by Propositions 3.6.5 and 3.6.7 we know that

$$\exp(x) = \omega^{\sum_{0<i<\mu} \omega^{g(\omega^{-i})}} = \omega^{\sum_{0<i<\mu} \omega^{-i}}$$

Since μ is an ε-number, for $i < \mu$, $\omega^{-i} \in \mathbf{No}_\mu$ but $\sum_{0<i<\mu} \omega^{-i} \notin \mathbf{No}_\mu$ (as a consequence of Theorem 3.3.28, the series having length μ, the length of the surreal number is at least μ). Therefore $x \in \mathbb{R}_\lambda^{\mathbf{No}_\mu}$ and $\exp x \notin \mathbb{R}_\lambda^{\mathbf{No}_\mu}$. □

The aim of Theorems 5.1.6 and 5.1.10 is to solve this problem, by proposing a new decomposition of $\mathbf{No}_\lambda$ as a union of fields that are stable under both exponential and logarithm.

5.1.3 Hierarchy of fields stable by exponential and logarithm

Definition 5.1.5. Let λ be an ε-number. Let α such that $\lambda = \varepsilon_\alpha$. We have

$$\lambda = \sup E_\lambda$$

where

$$E_\lambda = \begin{cases} \{\omega \uparrow\uparrow n \mid n \in \mathbb{N}\} & \alpha = 0 \\ \{\varepsilon_\beta \uparrow\uparrow n \mid n \in \mathbb{N}\} & \beta + 1 = \alpha \\ \{\varepsilon_\beta \mid \beta < \alpha\} & \alpha \in \mathbf{Lim} \setminus \{0\} \end{cases}$$

and $\uparrow\uparrow$ is the Knuth's arrow notation. Namely,

$$x \uparrow\uparrow 0 = 1 \qquad \text{and} \qquad \forall n \in \mathbb{N} \quad x \uparrow\uparrow (n+1) = x^{x \uparrow\uparrow n}$$

In other words,

$$x \uparrow\uparrow n = \left. x^{x^{\cdot^{\cdot^{\cdot^{x}}}}} \right\} n \text{ occurences of } x$$

We may write $E_\lambda = (e_\beta)_{\beta < \gamma_\lambda}$ where $\gamma_\lambda = \begin{cases} \omega & \beta + 1 = \alpha \text{ or } \alpha = 0 \\ \alpha & \alpha \in \mathbf{Lim} \setminus \{0\} \end{cases}$

Let Γ be an Abelian subgroup of $\mathbf{No}$. We denote by $\Gamma^{\uparrow\lambda}$ the family of group $(\Gamma_\beta)_{\beta < \gamma_\lambda}$ defined as follows :

- $\Gamma_0 = \Gamma$
- $\Gamma_{\beta+1}$ is the group generated by Γ_β, $\mathbb{R}_{e_\beta}^{g\left((\Gamma_\beta)^*_+\right)}$ and $\left\{ h(a_i) \;\middle|\; \sum\limits_{i<\nu} r_i\omega^{a_i} \in \Gamma_\beta \right\}$
- For limit ordinal numbers β, $\Gamma_\beta = \bigcup\limits_{\gamma<\beta} \Gamma_\gamma$.

In this section we prove the following:

Theorem 5.1.6. *Let Γ be an Abelian subgroup of* **No** *and λ be an ε-number, then $\mathbb{R}_\lambda^{\Gamma\uparrow\lambda}$ (see Definition 3.3.10) is stable under exponential and logarithmic functions.*

We will actually prove a stronger result which is the following proposition.

Proposition 5.1.7. *Let λ be an ε-number and $(\Gamma_i)_{i\in I}$ be a family of Abelian subgroups of* **No**. *Then $\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is stable under* $\exp$ *and* $\ln$ *if and only if*

$$\bigcup_{i\in I}\Gamma_i = \bigcup_{i\in I}\mathbb{R}_\lambda^{g\left((\Gamma_i)^*_+\right)}$$

Proof. (NC ⇒) We assume that $\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is stable under both exponential and logarithm. Then for any $x = \sum\limits_{i<\nu} r_i\omega^{a_i}$ a purely infinite number, we have

$$\exp x = \omega^{\sum\limits_{i<\nu} r_i\omega^{g(a_i)}} \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$$

and therefore
$$\sum_{i<\nu} r_i\omega^{g(a_i)} \in \bigcup_{j\in I}\Gamma_j$$

This being true for any family $(a_i)_{i<\nu}$ of Γ_j, for any $j\in I$. Hence, $\bigcup\limits_{i\in I}\mathbb{R}_\lambda^{g\left((\Gamma_i)^*_+\right)} \subseteq \bigcup\limits_{i\in I}\Gamma_i$.

Conversely, for any $j\in I$ and any $a\in\Gamma_j$ then we have $\ln\omega^a \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$. Writing $a = \sum\limits_{i<\nu} r_i\omega^{a_i}$, we get $\sum\limits_{i<\nu} r_i\omega^{h(a_i)} \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$. To say it another way,

$$\exists k\in I \quad \forall i<\nu \qquad h(a_i)\in\Gamma_k$$

or
$$\exists k\in I \quad \forall i<\nu \qquad a_i \in g\left((\Gamma_k)^*_+\right)$$

Then, there is some $k\in I$ such that $a\in\mathbb{R}_\lambda^{g\left((\Gamma_k)^*_+\right)}$. Hence, for all $j\in I$, $\Gamma_j \subseteq \bigcup\limits_{i\in I}\mathbb{R}_\lambda^{g\left((\Gamma_i)^*_+\right)}$. Finally,

$$\bigcup_{i\in I}\mathbb{R}_\lambda^{g\left((\Gamma_i)^*_+\right)} \supseteq \bigcup_{i\in I}\Gamma_i$$

Having both inclusions, we get
$$\bigcup_{i\in I}\mathbb{R}_\lambda^{g\left((\Gamma_i)^*_+\right)} = \bigcup_{i\in I}\Gamma_i$$

(SC ⇐) We assume that $\bigcup\limits_{i\in I}\mathbb{R}_\lambda^{g\left((\Gamma_i)^*_+\right)} = \bigcup\limits_{i\in I}\Gamma_i$. We distinguish the proof in several steps

(i) First take $x = \sum\limits_{i<\nu} r_i\omega^{a_i} \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ being appreciable, i.e. $a_i \leq 0$ for all $i<\nu$. By definition there is some $j\in I$ such that x is an element of $\mathbb{R}_\lambda^{\Gamma_j}$. Following Theorem 3.6.3,

$$\operatorname{supp}\exp x \subseteq \langle\operatorname{supp} x\rangle$$

where $\langle\operatorname{supp} x\rangle$ is the monoid generated by $\operatorname{supp} x$ in Γ_j. In particular $\operatorname{supp}\exp x\subseteq\Gamma_j$. Then, Proposition 2.4.5 ensures that the order type of $\operatorname{supp}\exp x$ is less than λ. Hence $\exp x\in\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$.

(ii) Let $x = \sum\limits_{i<\nu} r_i\omega^{a_i} \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ a purely infinite number. Let $j\in I$ such that $x\in\mathbb{R}_\lambda^{\Gamma_j}$ that is that $a_i\in(\Gamma_j)^*_+$ for all $i<\nu$. We have $\exp x = \omega^{\sum\limits_{i<\nu} r_i\omega^{g(a_i)}}$ and

$$\sum_{i<\nu} r_i\omega^{g(a_i)} \in \mathbb{R}_\lambda^{g\left((\Gamma_j)^*_+\right)}$$

By assumption, $\mathbb{R}_\lambda^{g\left((\Gamma_j)^*_+\right)} \subseteq \bigcup\limits_{i\in I}\Gamma_i$. Then $\exp x\in\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$.

(iii) We now make use of both Items (i) and (ii). Let $x \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ be arbitrary. Let x_∞ its purely infinite part and x_a its appreciable part. Then $x = x_\infty + x_a$ and $\exp x = \exp(x_\infty)\exp(x_a)$. Using (ii) and (i) respectively, we have $\exp x_\infty \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ and $\exp x_a \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$. Then since $\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is a field, $\exp x \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$.

(iv) Similarly to Point (i), if $x = \sum_{i<\nu} r_i\omega^{a_i} \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is infinitesimal, i.e. $a_i < 0$ for all $i < \nu$, then

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{x^k}{k} \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$$

(v) Let $a \in \bigcup_{i\in I}\Gamma_i$. By assumption there is $j \in I$ such that $a \in \mathbb{R}_\lambda^{g\left((\Gamma_j)_+^*\right)}$. Hence, we can write $a = \sum_{i<\nu} r_i\omega^{g(a_i)}$ where $\nu < \lambda$ and $a_i \in (\Gamma_j)_+^*$ for all $i < \nu$. Then, $\ln\omega^a = \sum_{i<\nu} r_i\omega^{a_i}$. Hence $\ln\omega^a \in \mathbb{R}_\lambda^{\Gamma_j} \subseteq \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$.

(vi) Let $x \in \left(\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}\right)_+^*$ be arbitrary and write it as $x = r\omega^a(1+\varepsilon)$ where ε is infinitesimal, r is a positive real number and a a surreal number. Then, $\ln x = \ln\omega^a + \ln r + \ln(1+\varepsilon)$. Then since $\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is a field, $\exp x \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$. Using (v) and (iv) respectively, we have $\ln\omega^a \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ and $\ln(1+\varepsilon) \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$. Then, since $\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is a field containing $\mathbb{R}$, $\ln x \in \mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$.

Item (iii) proves that $\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is stable under exponential and Item (vi) that $\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is stable under logarithm. This is what was announced.

□

We are now ready to prove the theorem. We use the notations of Definition 5.1.5.

Proof of Theorem 5.1.6. We write $\Gamma^{\uparrow\lambda} = (\Gamma_\beta)_{\beta<\gamma_\lambda}$. Using Proposition 5.1.7, we just need to show

$$\bigcup_{\beta<\gamma_\lambda}\Gamma_\beta = \bigcup_{\beta<\gamma_\lambda}\mathbb{R}_\lambda^{g\left((\Gamma_\beta)_+^*\right)}$$

(⊇) Let $x \in \mathbb{R}_\lambda^{g\left((\Gamma_\beta)_+^*\right)}$. Let $n < \gamma_\lambda$ minimal such that $\nu(x) < e_n$. Then $x \in \Gamma_{\max(n,\beta)}$.

(⊆) Let $x \in \Gamma_\beta$. Write $x = \sum_{i<\nu} r_i\omega^{a_i}$. We also have $x = \sum_{i<\nu} r_i\omega^{g(h(a_i))}$ and $h(a_i) \in \Gamma_{\beta+1}$. Then $x \in \mathbb{R}_\lambda^{g\left((\Gamma_{\beta+1})_+^*\right)}$.

□

Another consequence of Proposition 5.1.7 is also the following:

Corollary 5.1.8. *Let λ be an ε-number and Γ be an Abelian subgroup of* **No**. *Then $\mathbb{R}_\lambda^\Gamma$ is stable under* $\exp$ *and* $\ln$ *if and only if $\Gamma = \mathbb{R}_\lambda^{g(\Gamma_+^*)}$.*

This result is quite similar to Theorem 5.1.6 but in the very particular case where $\bigcup_{G\in\Gamma^{\uparrow\lambda}} G = \Gamma$. This applies for instance when $\Gamma = \{0\}$. In this case, we get $\mathbb{R}_\lambda^\Gamma = \mathbb{R}$. If λ is a regular cardinal we get another example considering $\mathbb{R}_\lambda^\Gamma = \Gamma = \mathbf{No}_\lambda$ which is a result that is very similar to the result of Kuhlmann and Shelah in [34].
We now try to make use of this result to give a decomposition of $\mathbf{No}_\lambda$ into an increasing union of subfield, each of them stable under exp and ln. To prove it, we first prove a proposition to ensure inclusion of the mentioned fields in the $\mathbf{No}_\lambda$.

Proposition 5.1.9. *Let λ be an ε-number and $\mu < \lambda$ an additive (or multiplicative) ordinal. If $\Gamma \subseteq \mathbf{No}_\mu$ then $\mathbb{R}_\lambda^{\Gamma^{\uparrow\lambda}} \subseteq \mathbf{No}_\lambda$*

Proof. Write $\Gamma^{\uparrow\lambda} = (\Gamma_\beta)_{\beta<\gamma_\lambda}$. What we have to prove is that for all $i < \gamma_\lambda$, $\Gamma_i \subseteq \mathbf{No}_{\mu_i}$ for some $\mu_i < \lambda$. We will even prove that $\mu_i = e_{k\oplus 2\otimes i}$ works for some fixed ordinal k. We prove it by induction on i.

- For $i = 0$, $\mu_0 = e_k$ with k the least ordinal such that $\mu \leq e_k$ works.
- Assume $i = j+1$ and that the property is true for j. Therefore Γ_i is the group generated by Γ_j, $\mathbb{R}_{e_j}^{g\left((\Gamma_j)_+^*\right)}$ and $\left\{h(a_k)\ \middle|\ \sum_{k<\nu} r_k\omega^{a_k} \in \Gamma_j\right\}$. Thanks to the induction hypothesis and Lemma 3.6.21, $g\left((\Gamma_j)_+^*\right) \subseteq \mathbf{No}_{\mu_j}$, since μ_j is an additive ordinal. Hence, thanks to Lemma 3.3.34, $\mathbb{R}_{e_j}^{g\left((\Gamma_j)_+^*\right)} \subseteq \mathbf{No}_{\omega^{\mu_j\otimes\omega}\otimes e_j}$. Finally, from Lemmas 3.6.23 and 3.3.34, $h(a_k) \in \mathbf{No}_{\omega^{\mu_j}}$. Thus, $\Gamma_i \subseteq \mathbf{No}_{\omega^{\mu_j\otimes\omega}\otimes e_j}$. Since $\omega^{\mu_j\otimes\omega}, e_j < e_{k\oplus 2\otimes i}$, and $e_{k\oplus 2\otimes i}$ is multiplicative, taking, $\mu_i = e_{k\oplus 2\otimes i}$ works.

- If $i < \gamma_\lambda$ is a limit ordinal, for all $j < i$, $\lambda > e_{k\oplus 2\otimes i} > e_{k\oplus 2\otimes j}$. Then, by the induction hypothesis on all $j < i$, $\Gamma_i \subseteq \mathbf{No}_{e_{k\oplus 2\otimes i}}$.

□

With the previous proposition, we have all what we need to prove the following:

Theorem 5.1.10. *Let λ be an ε-number. $\mathbf{No}_\lambda = \bigcup_\mu \mathbb{R}_\lambda^{\mathbf{No}_\mu{}^{\uparrow\lambda}}$, where μ ranges over the additive ordinals less than λ (equivalently, μ ranges over the multiplicative ordinals less λ),*

Proof. Using Theorem 3.4.6, we know that

$$\mathbf{No}_\lambda = \bigcup_{\mu\in\{\,\mu<\lambda\ \mid\ \mu \text{ additive ordinal}\}} \mathbb{R}_\lambda^{\mathbf{No}_\mu}$$

By definition of $\mathbb{R}_\lambda^{\mathbf{No}_\mu{}^{\uparrow\lambda}}$, it must contain $\mathbb{R}_\lambda^{\mathbf{No}_\mu}$ and then

$$\mathbf{No}_\lambda \subseteq \bigcup_{\mu\in\{\,\mu<\lambda\ \mid\ \mu \text{ additive ordinal}\}} \mathbb{R}_\lambda^{\mathbf{No}_\mu{}^{\uparrow\lambda}}$$

On the other hand, applying Proposition 5.1.9 gives

$$\bigcup_{\mu\in\{\,\mu<\lambda\ \mid\ \mu \text{ additive ordinal}\}} \mathbb{R}_\lambda^{\mathbf{No}_\mu{}^{\uparrow\lambda}} \subseteq \mathbf{No}_\lambda$$

and this concludes the proof. □

5.1.4 Strictness of the Hierarchy

The hierarchy in Theorem 5.1.10 is strict:

Theorem 5.1.11. *For all ε-number λ, the hierarchy in previous theorem is strict:*

$$\mathbb{R}_\lambda^{\mathbf{No}_\mu{}^{\uparrow\lambda}} \subsetneq \mathbb{R}_\lambda^{\mathbf{No}_{\mu'}{}^{\uparrow\lambda}}$$

for all multiplicative ordinals μ and μ' such that $\omega < \mu < \mu' < \lambda$.

This theorem comes from the following: we claim that the construction of $\Gamma^{\uparrow\lambda}$ does not create new log-atomic numbers (up to some iteration of $\exp$ or $\ln$). After that, we will prove that we do introduce new log-atomic numbers when going trough the hierarchy.

Lemma 5.1.12. *Write $\Gamma^{\uparrow\lambda} = (\Gamma_\beta)_{\beta<\gamma_\lambda}$, and let*

$$L = \left\{ \exp_n x, \ln_n x \;\middle|\; x \in \mathbb{L},\quad n \in \mathbb{N},\quad \exists y \in \mathbb{R}_\lambda^\Gamma\ \exists P \in \mathcal{P}(y)\ \exists k \in \mathbb{N}\quad P(k) = x \right\}$$

we have for all $i < \gamma_\lambda$,

$$L = \left\{ \exp_n x, \ln_n x \;\middle|\; x \in \mathbb{L},\quad n \in \mathbb{N},\quad \exists y \in \mathbb{R}_\lambda^{\Gamma_i}\ \exists P \in \mathcal{P}(y)\ \exists k \in \mathbb{N}\quad P(k) = x \right\}$$

Proof. We prove it by induction on i.

- For $i = 0$, $\Gamma_0 = \Gamma$ then there is noting to prove.
- Assume the property for some ordinal $i < \gamma_\lambda$. We prove it for $i + 1$.
  - ($\subseteq$) Trivial since $\mathbb{R}_\lambda^{\Gamma_i} \subseteq \mathbb{R}_\lambda^{\Gamma_{i+1}}$.
  - ($\supseteq$) Let $x \in \mathbb{L}$, $y \in \mathbb{R}_\lambda^{\Gamma_{i+1}}$, $P \in \mathcal{P}(y)$ and $k \in \mathbb{N}$ such that $P(k) = x$. Write $P(0) = r\omega^a$ a term of x with $a \in \Gamma_{i+1}$. Then a can be written

$$a = u + v + \sum_{j=1}^{k} \sigma_j h(w_j)$$

with $u \in \Gamma_i$, $v \in \mathbb{R}_{e_i}^{g\left((\Gamma_i)_+^*\right)}$, $\sigma_j \in \{-1, 1\}$ and $w_j \in \Gamma_i$. By definition of a path, $P(1)$ is a purely infinite term of

$$\ln\omega^a = \ln\omega^u + \ln\omega^v + \sum_{j=1}^{k} \ln\omega^{\sigma_j h(w_j)} = \ln\omega^u + \ln\omega^v + \sum_{j=1}^{k} \sigma_j \ln\omega^{h(w_j)}$$

Then, up to a real factor s, $P(1)$ is a term of either $\ln\omega^u$ or $\ln\omega^v$ or $\ln\omega^{h(w_j)}$ for some j.

$\because$ Case 1: $sP(1)$ is a purely infinite term of $\ln \omega^u$. Then the function

$$Q(m) = \begin{cases} \omega^u & \text{if } m = 0 \\ sP(1) & \text{if } m = 1 \\ P(m) & \text{if } m > 1 \end{cases}$$

is a path of $\omega^u \in \mathbb{R}_\lambda^{\Gamma_i}$. Then, if $m \geq \max(2, n)$, $Q(m) = P(m) = \ln_{m-n}(x)$ then for all $n \in \mathbb{N}$,

$$\exp_n x, \ln_n x \in \left\{ \exp_n x, \ln_n x \;\middle|\; \begin{array}{c} x \in \mathbb{L}, \quad n \in \mathbb{N}, \\ \exists y \in \mathbb{R}_\lambda^{\Gamma_{i+1}} \; \exists P \in \mathcal{P}(y) \; \exists k \in \mathbb{N} \quad P(k) = x \end{array} \right\}$$

$\because$ Case 2: $sP(1)$ is a purely infinite term of $\ln \omega^v$. Write $v = \sum_{i<\nu'} s_i \omega^{g(b_i)}$ where $b_i \in \Gamma_k$. Again, the function

$$Q(m) = \begin{cases} sP(1) & \text{if } m = 0 \\ P(m+1) & \text{if } m > 0 \end{cases}$$

is a path of $\ln \omega^v = \sum_{i<\nu'} s_i \omega^{b_i} \in \mathbb{R}_\lambda^{\Gamma_i}$. Then,if $m \geq \max(1, n-1)$, $Q(m) = \ln_{m-n+1} x \in L$ and we are done.

$\because$ Case 3: $sP(1)$ is a purely infinite term of $\ln \omega^{h(w_j)}$. From the definition of w_j, there is $s' \in \mathbb{R}^*$ such that $s'\omega^{w_j}$ is a term of some element of $y \in \Gamma_n$. Then $s'\omega^{h(w_i)}$ is a purely infinite term of $\ln \omega^y$. Then the function

$$Q(m) = \begin{cases} \omega^y & \text{if } m = 0 \\ s'\omega^{h(w_i)} & \text{if } m = 1 \\ sP(1) & \text{if } m = 2 \\ P(m-1) & \text{if } m > 2 \end{cases}$$

is a path of $\omega^y \in \mathbb{R}_\lambda^{\Gamma_i}$. Then, if $m \geq \max(3, n+1)$, $Q(m) = \ln_{m-n-1} x \in L$ and we are done.

- Let $i < \gamma_\lambda$ be a limit ordinal. Assume the property for $j < i$. We have that $\Gamma_i = \bigcup_{j<i} \Gamma_j$. Again we just need to prove one inclusion, the other one being trivial. Let $x \in \mathbb{L}$ and $y \in \mathbb{R}_\lambda^{\Gamma_i}$, $P \in \mathcal{P}(y)$ and $n \in \mathbb{N}$ minimal such that $P(n) = x$. Write $P(0) = r\omega^a$ with $a \in \Gamma_i$. Then there is $j < i$ such that $a \in \Gamma_j$. In particular P is a path of $r\omega^a \in \mathbb{R}_\lambda^{\Gamma_j}$. We conclude using induction hypothesis on j.

$\square$

Corollary 5.1.13. *Let Γ be an abelian additive subgroup of* **No** *and*

$$L = \left\{ \exp_n x, \ln_n x \;\middle|\; x \in \mathbb{L}, \quad n \in \mathbb{N}, \quad \exists y \in \mathbb{R}_\lambda^\Gamma \; \exists P \in \mathcal{P}(y) \; \exists k \in \mathbb{N} \quad P(k) = x \right\}$$

Then,

$$L = \left\{ \exp_n x, \ln_n x \;\middle|\; \begin{array}{c} x \in \mathbb{L}, \quad n \in \mathbb{N}, \\ \exists y \in \mathbb{R}_\lambda^{\Gamma^{\uparrow\lambda}} \; \exists P \in \mathcal{P}(y) \; \exists k \in \mathbb{N} \quad P(k) = x \end{array} \right\}$$

Proof. Just apply the definition of $\mathbb{R}_\lambda^{\Gamma^{\uparrow\lambda}}$ and Lemma 5.1.12.

$\square$

We now prove the Theorem 5.1.11.

Proof of Theorem 5.1.11. Let λ be an epsilon number. Let $\mu < \mu' < \lambda$ be multiplicative ordinals. Let $x = \omega^{\omega^{-\mu}}$. Clearly, $x \in \mathbb{R}_\lambda^{\mathbf{No}_{\mu'}} \subseteq \mathbb{R}_\lambda^{\mathbf{No}_{\mu'}{}^{\uparrow\lambda}}$. So we will prove that $x \notin \mathbb{R}_\lambda^{\mathbf{No}_\mu{}^{\uparrow\lambda}}$. Note x is a log-atomic number, indeed using Corollary 3.6.22 $\ln_n x = \omega^{\omega^{-\mu-n}}$. Then applying Corollary 5.1.13 to both $\mathbf{No}_\mu$ and $\mathbf{No}_{\mu'}$ we just need to show that

$$x \notin \left\{ \exp_n x, \ln_n x \;\middle|\; x \in \mathbb{L}, \quad n \in \mathbb{N}, \quad \exists y \in \mathbb{R}_\lambda^{\mathbf{No}_\mu} \; \exists P \in \mathcal{P}(y) \; \exists k \in \mathbb{N} \quad P(k) = x \right\}$$

Assume the converse. Then there is some path P such that $P(0) \in \mathbb{R}\omega^{\mathbf{No}_\mu}$ and there is some natural numbers $n, k \in \mathbb{N}$ such that $P(k) = \ln_n x$. We prove by induction on i that for all $i \in [\![0 \; ; \; k]\!]$, $|a_i|_{+-} \geq \mu$ where $P(i) = r_i \omega^{a_i}$,

- For $i = k$, $P(i) = \omega^{\omega^{-\mu-n}}$ and using Theorem 3.3.28,

$$\left|\omega^{-\mu-n}\right|_{+-} = \omega \otimes (\mu + n) \geq \mu$$

- Assume the property for some $i \in [\![1\,;\,k]\!]$. By definition of a path, writing $P(i-1) = r_{i-1}\omega^{a_{i-1}}$ and $a_{i-1} = \sum_{j<\nu} s_j \omega^{b_j}$, there is some $j_0 < \nu$ such that $b_{j_0} = g(a_i)$ and $s_{j_0} = r_i$. Using induction hypothesis and Corollary 3.6.26, $\left|\omega^{b_{j_0}}\right|_{+-} \geq \mu$ and therefore $\left|s_{j_0}\omega^{b_{j_0}}\right|_{+-} \geq \mu$. Now using Lemma 3.3.35, $|a_{i-1}|_{+-} \geq \left|s_{j_0}\omega^{b_{j_0}}\right|_{+-} \geq \mu$.

The induction principle conclude that $|a_0|_{+-} \geq \mu$. But since $P(0) \in \mathbb{R}\omega^{\mathbf{No}_\mu}$, $|a_0|_{+-} < \mu$. We reach a contradiction. Then $x \notin \mathbb{R}_\lambda^{\mathbf{No}_\mu}$. □

5.2 Controlling the length of the series of anti-derivatives

In order to determine how surreal fields can be stable under the anti-derivative, which is a very complicated operation, we need to have the control over the length of the series of the anti-derivatives of surreal numbers. This section is about tackling this problem.

5.2.1 Case disjunction on special cases

In this first subsection, we tackle some special cases to go more and more general. Namely, we look at the cases involved in the asymptotic anti-derivative as we have already seen them in Corollary 3.10.10 or in Definition 3.10.13. Recall that we can see a monomial ω^a as $\omega^a = \partial u \exp \varepsilon$ where $u = \ln_n \kappa_{-\alpha}$ for some $n \in \mathbb{N}$, some ordinal α and some surreal $\varepsilon \prec u$. Taking such a decomposition with (α, n) maximal for the lexicographic order, we dissociate two cases: $\varepsilon \preceq \ln u$ and $\varepsilon \succ \ln u$. Note that it is impossible that $\varepsilon \sim -\ln u$ without breaking the maximality of (α, n). Note also that this maximal element exists since a close look at ∂ shows that (α, n) is in fact the least element such that

$$-\sum_{(\beta,m)<\alpha,n} \ln_m \kappa_{-\beta} \not\trianglelefteq \ln x$$

Case $\varepsilon \preceq \ln u$

We first look at monomials ω^a and determine all the terms that can appear in the series of $\Phi(\omega^a)$.

Lemma 5.2.1. *Assume $x = \omega^a = \partial u \exp \varepsilon$ with $\varepsilon = r \ln u + \eta$ and $u = \ln_n \kappa_{-\alpha}$. Let $b \in \operatorname{supp} \Phi(\omega^a)$. Then, there is a path $P \in \mathcal{P}_{\mathbb{L}}(\eta)$ such that*

$$\omega^b \asymp \partial u \exp \left(r \ln u + \eta - \sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta > \alpha,\, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{i=0}^{+\infty} \ln |P(i)| \right)$$

Proof. It is just a calculation. First notice that $\dfrac{\omega^a}{r+1}\dfrac{u}{\partial u}$ is a term as a product of terms. Then, let $b \in \operatorname{supp} \Phi(\omega^a)$. There is path P of η such that

$$\omega^b \asymp \omega^a \frac{u}{\partial u} \partial P = u \partial P \exp(r \ln u + \eta)$$

write

$$\partial P = P(0) \cdots P(k_P - 1) \partial_{\mathbb{L}} P(k_P)$$

Since $P(0)$ is a term of $\eta \prec \ln u$, we also have $P(0) \prec \ln u$. Moreover since η consists in purely infinite term, so is $P(0)$ and then $\ln |P(0)| \prec P(0)$. Since $P(1)$ is a purely infinite term of $\ln |P(0)|$, we get that $P(1) \prec P(0)$. By induction, for all i, $P(i+1) \preceq P(i) \leq P(0)$. In particular, $P(k_P) \preceq P(0) \preceq^k \kappa_{-\alpha}$. Then, $\kappa_{-\alpha} \succeq^K P(k_P)$. That leads to

$$\partial_{\mathbb{L}}(P(k_P)) = \exp\left(-\sum_{\beta\leq\alpha,\ m\in\mathbb{N}^*} \ln_m \kappa_{-\beta} - \sum_{\substack{\beta>\alpha, m\in\mathbb{N}^* \\ \beta \mid \kappa_{-\beta}\succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{m=1}^{+\infty} \ln_m P(k_P)\right)$$

$$\partial_{\mathbb{L}}(P(k_P)) = \partial u \exp\left(-\sum_{m=n+1}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta>\alpha, m\in\mathbb{N}^* \\ \beta \mid \kappa_{-\beta}\succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{m=1}^{+\infty} \ln_m P(k_P)\right)$$

Since $P(k_P) \in \mathbb{L}$,

$$\partial_{\mathbb{L}}(P(k_P)) = \partial u \exp\left(-\sum_{m=n+1}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta>\alpha, m\in\mathbb{N}^* \\ \beta \mid \kappa_{-\beta}\succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{i=k_P}^{+\infty} \ln|P(i)|\right)$$

Then

$$\partial P = \partial u \exp\left(-\sum_{m=n+1}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta>\alpha, m\in\mathbb{N}^* \\ \beta \mid \kappa_{-\beta}\succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{i=0}^{+\infty} \ln|P(i)|\right)$$

Finally,

$$\omega^b \asymp \partial u \exp(r\ln u + \eta) u \exp\left(-\sum_{m=n+1}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta>\alpha, m\in\mathbb{N}^* \\ \beta \mid \kappa_{-\beta}\succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{i=0}^{+\infty} \ln|P(i)|\right)$$

$$\omega^b \asymp \partial u \exp\left(r\ln u + \eta - \sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta>\alpha, m\in\mathbb{N}^* \\ \beta \mid \kappa_{-\beta}\succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{i=0}^{+\infty} \ln|P(i)|\right)$$

□

We can now look at what appear in all the iterations of Φ.

Proposition 5.2.2. *Assume $x = \omega^a = \partial u \exp\varepsilon$ with $\varepsilon = r\ln u + \eta$ and $u = \ln_n \kappa_{-\alpha}$. We denote for $P_0,\ldots,P_k \in \mathcal{P}_{\mathbb{L}}(\eta)$ and $i_1,\ldots,i_k \in \mathbb{N}^*$,*

$$e(P_0,\ldots,P_k; i_1,\ldots,i_k) = -(k+1)\sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{j=0}^{k} \sum_{\substack{\beta>\alpha, m\in\mathbb{N}^* \\ \beta \mid \kappa_{-\beta}\succeq^K P_j(k_{P_j})}} \ln_m \kappa_{-\beta} + \sum_{j=0}^{k}\sum_{i=i_j}^{+\infty} \ln|P_j(i)|$$

with $i_0 = 0$. Let

$$E_{1,k} = \left\{ e(P_0,\ldots,P_k; i_1,\ldots,i_k) \;\middle|\; \begin{array}{c} P_0,\ldots,P_k \in \mathcal{P}_{\mathbb{L}}(\eta) \\ i_0 = 0 \quad i_1,\ldots,i_k \in \mathbb{N}^* \\ \forall j \in [\![1\,;\,k]\!] \quad \exists j' \in [\![0\,;\,j-1]\!] \quad \forall i \in [\![0\,;\,i_j - 1]\!] \qquad P_{j'}(i) = P_j(i) \\ \forall j \in [\![1\,;\,k]\!] \quad \operatorname{supp} P_j(i_j) \subseteq \operatorname{supp} e(P_0,\ldots,P_{k-1}; i_1,\ldots,i_{k-1}) \end{array} \right\}$$

$$E_1 = \bigcup_{k\in\mathbb{N}} E_{1,k}$$

$$E_2 = \left\{ -\sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\gamma<\beta,\ m\in\mathbb{N}^*} \ln_m \kappa_{-\gamma} - \sum_{m=1}^{p} \ln_m \kappa_{-\beta} \;\middle|\; \exists P \in \mathcal{P}_{\mathbb{L}}(\eta) \begin{array}{c} \beta > \alpha \\ \kappa_{-\beta} \succeq^K P(k_P) \\ p \in \mathbb{N} \end{array} \right\}$$

$$E_3 = \left\{ -\sum_{m=n+2}^{p} \ln_m \kappa_{-\alpha} \;\middle|\; p \geq n+2 \right\}$$

$$E = E_1 \cup E_2 \cup E_3$$

and $\langle E\rangle$ be the monoid it generates. Let $b \in \bigcup_{\ell=0}^{+\infty} \operatorname{supp} \Phi^{\ell}(\omega^a)$. Then, there is $y \in \langle E\rangle$ such that

$$\omega^b \asymp \partial u \exp(r\ln u + \eta + y)$$

Proof. We prove it by induction on ℓ.

- If $b \in \operatorname{supp}\omega^a$, then $y = 0$ works.

- Assume the property for $\ell \in \mathbb{N}$ and let $b \in \operatorname{supp}\Phi^{\ell+1}(\omega^a)$. Then there is $c \in \operatorname{supp}\Phi^{\ell}(\omega^a)$ such that $b \in \operatorname{supp}\Phi(\omega^c)$. Apply the induction hypothesis on c and on y associated to c. Since any element $e \in E$ is such that $e \prec \ln u$, we have $y \prec \ln u$ then Apply Lemma 5.2.1 to get that there is $P \in \mathcal{P}_{\mathbb{L}}(\eta + y)$ such that

$$\omega^b \asymp \partial u \exp\left(r\ln u + \eta + y - \sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta > \alpha, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{i=0}^{+\infty} \ln|P(i)| \right)$$

If $P(0)$ a term of η, up to some real factor, then there is a real number s and some $e \in E_{1,0}$ such that

$$\exp\left(- \sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta > \alpha, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{i=0}^{+\infty} \ln|P(i)| \right) = s \exp e$$

Then $y + e \in \langle E \rangle$ and $\omega^b \asymp \partial u \exp(r\ln u + \eta + y + e)$. If not, then $P(0)$ is a term of y (not up to a real factor, an actual term). Hence, we have the following cases :

➢ $P(0) = s\ln_p \kappa_{-\alpha}$ for some $s \in \mathbb{R}^*_-$ and $p \geq n+2$. Then,

$$- \sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta > \alpha, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{i=0}^{+\infty} \ln|P(i)| = \ln|s| - \sum_{m=n+2}^{p} \ln_m \kappa_{-\alpha} \in \ln|s| + E_3$$

Then $$y - \sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta > \alpha, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{i=0}^{+\infty} \ln|P(i)| \in \mathbb{R} + \langle E \rangle$$

➢ $P(0) = s\ln_p \kappa_{-\beta}$ with $\beta > \alpha$ and $p \in \mathbb{N}^*$ such that there is some path $Q \in \mathcal{P}_{\mathbb{L}}(\eta)$ such that $\kappa_{-\beta} \succeq^K Q(k_Q)$. Then

$$- \sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta > \alpha, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{i=0}^{+\infty} \ln|P(i)| \in \ln|s| + E_2$$

Then $$y - \sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta > \alpha, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{i=0}^{+\infty} \ln|P(i)| \in \mathbb{R} + \langle E \rangle$$

➢ There are some paths $P_0, \ldots, P_k \in \mathcal{P}_{\mathbb{L}}(\eta)$ and some non-zero integers $i_1, \ldots, i_k$ such that

$$\forall j \in [\![1 \,;\, k]\!] \quad \exists j' \in [\![0 \,;\, j-1]\!] \quad \forall i \in [\![0 \,;\, i_j - 1]\!] \qquad P_{j'}(i) = P_j(i)$$

and

$$\exists y' \in \langle E \rangle \qquad y = y' - (k+1)\sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{j=0}^{k} \sum_{\substack{\beta > \alpha, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P_j(k_{P_j})}} \ln_m \kappa_{-\beta} + \sum_{j=0}^{k} \sum_{i=i_j}^{+\infty} \ln|P_j(i)|$$

and such that $P(0) \in \mathbb{R}z$ for some z a term of some $\ln|P_j(i_{k+1}')|$ with $j \in [\![0 \,;\, k]\!]$ and $i_{k+1}' \geq i_j$. Let P_{k+1} be the following path :

$$P_{k+1}(i) = \begin{cases} P_j(i) & i \leq i_{k+1}' \\ z & i = i_{k+1}' + 1 \\ P(i - i_{k+1}' - 1) & i > i_{k+1}' + 1 \end{cases}$$

Then, $P_{k+1} \in \mathcal{P}(\eta)$. Moreover, $\partial P_{k+1} = \underbrace{P_j(0)\cdots P_j(i_{k+1}')}_{\neq 0}\underbrace{\partial P}_{\neq 0}$. Then $P_{k+1} \in \mathcal{P}_{\mathbb{L}}(\eta)$. Note also that for all β,

$$\kappa_{-\beta} \succeq^K P_{k+1}(k_{P_{k+1}}) \Longleftrightarrow \kappa_{-\beta} \succeq^K P(k_P)$$

Finally,

$$\begin{aligned} & - \sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta > \alpha, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{i=0}^{+\infty} \ln|P(i)| \\ = & - \sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta > \alpha, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P_{k+1}(k_{P_{k+1}})}} \ln_m \kappa_{-\beta} + \sum_{i=i_{k+1}'+1}^{+\infty} \ln|P_{k+1}(i)| + \underbrace{\ln\left|\frac{P(0)}{z}\right|}_{\in \mathbb{R}^*_+} \end{aligned}$$

From that we derive that

$$\begin{aligned} & y - \sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta > \alpha, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{i=0}^{+\infty} \ln |P(i)| \\ = {} & y' - (k+2) \sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{j=0}^{k+1} \sum_{\substack{\beta > \alpha, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P_j(k_{P_j})}} \ln_m \kappa_{-\beta} + \sum_{j=0}^{k+1} \sum_{i=i_j}^{+\infty} \ln |P_j(i)| + \ln \left| \frac{P(0)}{z} \right| \\ \in {} & \mathbb{R} + \langle E \rangle \end{aligned}$$

where $i_{k+1} = {i_{k+1}}' + 1$ and $P_{k+1}(i_k) = z$ has indeed its support (which is reduced to a singleton) included in the one of $e(P_0, \ldots, P_k; i_1, \ldots, i_k)$.

Then there is a real number s, and $e \in \langle E \rangle$ such that

$$\omega^b \asymp \partial u \exp(r \ln u + \eta + e + s) \asymp \partial u \exp(r \ln u + \eta + e)$$

Then we get the property at rank $\ell + 1$.

By the induction principle, we conclude that the proposition is true for any $\ell \in \mathbb{N}$. ☐

Before looking at the general case, we now look at the case where all the monomial share the same u and the same first term of ε.

Corollary 5.2.3. *Let $x = \sum_{i<\nu} r_i \omega^{a_i}$ such that*

$$\exists u = \ln_n \kappa_{-\alpha} \quad \exists r \in \mathbb{R} \quad \forall a \in \operatorname{supp} x \quad \exists \eta \prec \ln u \qquad \omega^a = \partial(u) \exp(r \ln u + \eta)$$

We denote for $P_0, \ldots, P_k \in \mathcal{P}_{\mathbb{L}}(x)$ and $i_1, \ldots, i_k \in \mathbb{N} \setminus \{0, 1\}$,

$$e(P_0, \ldots, P_k; i_1, \ldots, i_k) = -(k+1) \sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{j=0}^{k} \sum_{\substack{\beta > \alpha, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P_j(k_{P_j})}} \ln_m \kappa_{-\beta} + \sum_{j=0}^{k} \sum_{i=i_j}^{+\infty} \ln |P_j(i)|$$

with $i_0 = 0$. Let

$$E_{1,k} = \left\{ e(P_0, \ldots, P_k; i_1, \ldots, i_k) \;\middle|\; \begin{array}{c} P_0, \ldots, P_k \in \mathcal{P}_{\mathbb{L}}(x) \\ \forall i \in [\![0 \,;\, k]\!] \qquad P_i(1) \prec \ln u \\ i_0 = 0 \quad i_1, \ldots, i_k \in \mathbb{N} \setminus \{0, 1\} \\ \forall j \in [\![1 \,;\, k]\!] \quad \exists j' \in [\![0 \,;\, j-1]\!] \quad \forall i \in [\![0 \,;\, i_j - 1]\!] \qquad P_{j'}(i) = P_j(i) \\ \forall j \in [\![1 \,;\, k]\!] \quad \operatorname{supp} P_j(i_j) \subseteq \operatorname{supp} e(P_0, \ldots, P_{k-1}; i_1, \ldots, i_{k-1}) \end{array} \right\}$$

$$E_1 = \bigcup_{k \in \mathbb{N}} E_{1,k}$$

$$E_2 = \left\{ -\sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\gamma < \beta,\, m \in \mathbb{N}^*} \ln_m \kappa_{-\gamma} - \sum_{m=1}^{p} \ln_m \kappa_{-\beta} \;\middle|\; \exists P \in \mathcal{P}_{\mathbb{L}}(x) \begin{array}{c} \beta > \alpha \\ \kappa_{-\beta} \succeq^K P(k_P) \\ p \in \mathbb{N} \end{array} \right\}$$

$$E_3 = \left\{ -\sum_{m=n+2}^{p} \ln_m \kappa_{-\alpha} \;\middle|\; p \geq n+2 \right\}$$

$$E = E_1 \cup E_2 \cup E_3$$

and $\langle E \rangle$ be the monoid it generates. Let $b \in \bigcup_{\ell=0}^{+\infty} \operatorname{supp} \Phi^\ell(x)$. Then, there is $y \in \langle E \rangle$ such that

$$\omega^b \asymp \exp(y)$$

Proof. Since Φ is strongly linear, we just need to apply Proposition 5.2.2 to each term of x. ☐

Proposition 5.2.4. *Let $x = \sum_{i<\nu} r_i \omega^{a_i}$ such that*

$$\exists u = \ln_n \kappa_{-\alpha} \quad \exists r \in \mathbb{R} \quad \forall a \in \operatorname{supp} x \quad \exists \eta \prec \ln u \qquad \omega^a = \partial(u) \exp(r \ln u + \eta)$$

Consider E_1, E_2 and E_3 as defined Corollary 5.2.3. Let γ be the smallest ordinal such that $\kappa_{-\gamma} \prec^K P(k_P)$ for all path $P \in \mathcal{P}_{\mathbb{L}}(\eta)$. Let λ the least ε-number greater than $\operatorname{NR}(x)$ and γ. Then $E = E_1 \cup E_2 \cup E_3$ is reverse well-ordered with order type at most $2\lambda + \omega(\gamma + 1)$.

Proof. First notice that E_3 is reverse well-ordered with order type ω. E_2 is also reverse well-ordered with order at most $\omega + \omega \otimes \gamma + n \leq \omega \otimes (\gamma + 1)$. We then focus on E_1. We denote again

$$e(P_0, \ldots, P_k; i_1, \ldots, i_k) = -(k+1) \sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{j=0}^{k} \sum_{\substack{\beta > \alpha, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P_j(k_{P_j})}} \ln_m \kappa_{-\beta} + \sum_{j=0}^{k} \sum_{i=i_j}^{+\infty} \ln |P_j(i)|$$

(i) We first claim that for all $i \geq 3$ and all path $P \in \mathcal{P}(x)$ such that $P(1) \prec \ln u$, $P(i) \prec P(2) \preceq \ln_2 u$. Let $P \in \mathcal{P}(x)$ such that $P(1) \prec \ln u$. Assume $P(2) \succ \ln_2 u$. Then, since $P(2)$ is a term of $\ln|P(1)|$, we also have $\ln|P(1)| \succ \ln_2(u)$. Then, either $\ln|P(1)| < -m\ln_2 u$ for all $m \in \mathbb{N}$, or $\ln|P(1)| > m\ln_2(u)$ for all $m \in \mathbb{N}$. By definition, $P(1)$ is purely infinite. In particular, $\ln|P(1)|$ cannot be negative. Then,

$$\forall m \in \mathbb{N} \qquad \ln|P(1)| > m\ln_2 u$$

and $$\forall m \in \mathbb{N} \qquad |P(1)| > (\ln u)^m \qquad \text{(exp is increasing)}$$

which is a contradiction with $P(1) \prec \ln u$, since $\ln u$ is infinitely large. Since, for $i \geq 2$, $P(i)$ is infinitely large, $\ln|P(i)| \prec P(i)$, and since $P(i+1) \preceq \ln|P(i)|$, we have for all $i \geq 1$, $P(i+1) \prec P(i)$. By induction, we get

$$\forall i \geq 3 \qquad P(i) \prec P(2) \preceq \ln_2 u$$

(ii) We claim that for all path $P \in \mathcal{P}(x)$ such that $P(1) \prec \ln u$, if $P(2) \asymp \ln_2 u$, then, denoting r the real number such that $P(2) \sim r\ln_2 u$, we have $0 < r \leq 1$. Let $P \in \mathcal{P}(x)$ such that $P(1) \prec \ln u$ and assume $P(2) \asymp \ln_2 u$. Since $P(2)$ is a term there is a non-zero real number r such that $P(1) = r\ln_2 u$. From (i), we know that $P(2)$ is the dominant term of $\ln|P(1)|$ so that

$$\ln|P(1)| \sim r\ln_2 u$$

If $r < 0$, Proposition 3.6.5 ensures that $|P(1)| \prec 1$ what is impossible since $P(1)$ is infinite. Then $r > 0$. If now $r > 1$ then again with Proposition 3.6.5, $|P(1)| \succ \ln u$ what is not true. Then, $0 < r \leq 1$.

(iii) For all j and $i \geq 2$, $\ln|P_j(i)| \preceq \ln_3 u \prec \ln_2 u$. Indeed, using (i), we know that $P_j(i) \preceq \ln_2 u$. Then, there is a natural number $m \geq 1$ such that $|P_j(i)| \leq m\ln_2 u$. Using the fact that ln is increasing,

$$\ln|P_j(i)| \leq \ln_3 u + \ln m \preceq \ln_3 u \prec \ln_2 u$$

(iv) We now claim that $E_{1,k} > E_{1,k+2}$. Indeed, using (ii) and (iii) if $e_1 \in E_{1,k}$, then there is $s \in [\,-(k+1)\,;-k\,]$ such that $e_1 \sim s\ln_2 u$. Similarly, for $e_2 \in E_{1,k+2}$, there is $s' \in [\,-(k+3)\,;-(k+2)\,]$ such that $e_2 \sim s'\ln_2 u$.

(v) We define the following sequence :

- $a_0 = \omega^{\omega^{\omega(\mathrm{NR}(x)+1)}}$
- $a_{k+1} = \omega^{\omega^{\omega(\omega(\mathrm{NR}(x)+\gamma+4)a_k+1)}}$

We show that $E_{1,k}$ is reverse well-ordered with order type less than a_k. We also claim that the equivalence classes of $E_{1,k}/\asymp$ are finite and that

$$\mathrm{NR}\left(\sum_{t\in E_{1,k}} \exp t\right) \leq \omega(\mathrm{NR}(x)+\gamma+4)a_k$$

We show it by induction on $k \in \mathbb{N}$.

- For $k = 0$, let $t \in E_{1,0}$. Take $P \in \mathcal{P}_{\mathbb{L}}(x)$, minimal for $<_{lex}$ such that $P(1) \prec \ln u$ and $t = e(P;)$. Then

$$\partial(\ln u)\exp t = |P(0)\cdots P(k_P-1)|\exp\left(-\sum_{\substack{\beta \mid \kappa_{-\beta} \succeq^K P(k_P)\\ m\in\mathbb{N}^*}} \ln_m \kappa_{-\beta} + \sum_{i=k_P}^{+\infty} \ln|P(i)|\right) = |\partial P|$$

Since there are finitely many paths $Q \in \mathcal{P}_{\mathbb{L}}(x)$ such that $\partial P \asymp \partial Q$, there are finitely many $t' \in E_{1,0}$ such that

$$\partial(\ln u)\exp t \asymp \partial(\ln u)\exp t'$$

Since exp is an increasing function and $\partial(\ln u) > 0$, we get , using Proposition 3.9.29, that $E_{1,0}$ is reverse well-ordered with order type less than $\omega \otimes \omega^{\omega^{\omega(\mathrm{NR}(x)+1)}} = \omega^{\omega^{\omega(\mathrm{NR}(x)+1)}} = a_0$. Finally, it remains to compute the nested rank of $\sum_{t\in E_{1,0}} \exp t$. Write

$$t = -\sum_{m=n+2}^{+\infty} \ln_m \kappa_{-\alpha} - \sum_{\substack{\beta > \alpha,\, m\in\mathbb{N}^*\\ \beta \mid \kappa_{-\beta} \succeq^K P_0(k_{P_0})}} \ln_m \kappa_{-\beta} + \sum_{i=0}^{+\infty} \ln|P_0(i)|$$

$$\begin{aligned}
\mathrm{NR}\,(t) &= \mathrm{NR}\left(-\sum_{m=n+2}^{+\infty}\ln_m\kappa_{-\alpha} - \sum_{\substack{\beta>\alpha,m\in\mathbb{N}^*\\ \beta\mid\kappa_{-\beta}\succeq^K P_0(k_{P_0})}}\ln_m\kappa_{-\beta} + \sum_{i=0}^{+\infty}\ln|P_0(i)|\right)\\
&\leq \mathrm{NR}\left(-\sum_{m=n+2}^{+\infty}\ln_m\kappa_{-\alpha} - \sum_{\substack{\beta>\alpha,m\in\mathbb{N}^*\\ \beta\mid\kappa_{-\beta}\succeq^K P_0(k_{P_0})}}\ln_m\kappa_{-\beta} + \sum_{i=k_{P_0}}^{+\infty}\ln|P_0(i)|\right)\\
&\quad+\sum_{i=0}^{k_{P_0}-1}\mathrm{NR}\,(\ln|P_0(i)|)+k_{P_0} && \text{(Lemma 3.8.24)}\\
&\leq (\omega\oplus\omega\otimes\gamma\oplus\omega)+\sum_{i=0}^{k_{P_0}-1}\mathrm{NR}\,(\ln|P_0(i)|)+k_{P_0} && \text{(Lemma 3.8.19)}\\
&\leq (\omega\oplus\omega\otimes\gamma\oplus\omega)+k_{P_0}(\mathrm{NR}(x)+1) && \text{(using Proposition 3.8.23)}\\
&\leq \omega(\mathrm{NR}(x)+\gamma+4)
\end{aligned}$$

Then, since the equivalence classes of $E_{1,0}/\asymp$ are finite,

$$\mathrm{NR}\left(\sum_{t\in E_{1,0}}\exp t\right)\leq\omega(\mathrm{NR}(x)+\gamma+4)a_0$$

- Assume the property for some $k\in\mathbb{N}$. Let $t\in E_{1,k+1}$. Let $(P_0,0),\dots,(P_{k+1},i_{k+1})$ minimal for the order $(<_{lex},<)_{lex}$ such that $t=e(P_0,\dots,P_{k+1};i_1,\dots,i_{k+1})$. Then,

$$t=e(P_0,\dots,P_k;i_1,\dots,i_k)-\sum_{m=n+2}^{+\infty}\ln_m\kappa_{-\alpha}-\sum_{\substack{\beta>\alpha,m\in\mathbb{N}^*\\ \beta\mid\kappa_{-\beta}\succeq^K P_{k+1}(k_{P_{k+1}})}}\ln_m\kappa_{-\beta}+\sum_{i=i_{k+1}}^{+\infty}\ln|P_{k+1}(i)|$$

Write $s=e(P_0,\dots,P_k;i_1,\dots,i_k)$. We then have,

$$\partial(\ln u)\exp t=\exp(s)\exp\left(-\sum_{\substack{\beta\mid\kappa_{-\beta}\succeq^K P_{k+1}(k_{P_{k+1}})\\ m\in\mathbb{N}^*}}\ln_m\kappa_{-\beta}+\sum_{i=i_{k+1}}^{+\infty}\ln|P_{k+1}(i)|\right)$$

Consider the following path : $\begin{cases}R(0)=\exp s\\ R(i)=P_{k+1}(i-1+i_{k+1}) & i>0\end{cases}$

It is indeed a path since, by definition of $E_{1,k+1}$, $\operatorname{supp}P_{k+1}(i_{k+1})$ must be contained in $\operatorname{supp}s$. Then,

$$\partial(\ln u)\exp t=\partial R$$

Moreover, $R\in\mathcal{P}_{\mathbb{L}}\left(\sum_{s\in E_{1,k}}\exp s\right)$. By induction hypothesis and Proposition 3.9.29, $E_{1,k+1}$ has order type less than

$$\omega^{\omega^{\omega(\omega(\mathrm{NR}(x)+\gamma+4)a_k+1)}}=a_{k+1}$$

Since the equivalences classes of $\mathcal{P}_{\mathbb{L}}\left(\sum_{s\in E_{1,k}}\exp s\right)/\asymp$ are finite, the ones of $E_{1,k+1}/\asymp$ are also finite. Finally, using Lemmas 3.8.24 and 3.8.19,

$$\begin{aligned}
\mathrm{NR}(t)&\leq(\omega\oplus\omega\otimes\gamma\oplus\omega)+\sum_{j=0}^{k+1}\sum_{i=i_j}^{k_{P_j}-1}\mathrm{NR}\,(\ln|P_j(i)|)+\sum_{j=0}^{k+1}\max(0,k_{P_j}-i_j)\\
&\leq\omega(\mathrm{NR}(x)+\gamma+4)
\end{aligned}$$

Then, $$\mathrm{NR}\left(\sum_{t'\in E_{1,k+1}}\exp t'\right)\leq\omega(\mathrm{NR}(x)+\gamma+4)a_{k+1}$$

We conclude thanks to the induction principle.

(vi) By easy induction, for all $k\in\mathbb{N}$, $a_k<\lambda$.

(vii) Using (iv), we get that for all $N\in\mathbb{N}$, $\bigcup_{k=0}^{N}E_{1,2k}$ is an initial segment of $\bigcup_{k\in\mathbb{N}}E_{1,2k}$. We also have that $\bigcup_{k=0}^{N}E_{1,2k+1}$ is an initial segment of $\bigcup_{k\in\mathbb{N}}E_{1,2k+1}$. Using (v), we get that $\bigcup_{k\in\mathbb{N}}E_{1,2k}$ has order type at most

$$\sup \left\{ \bigoplus_{k=0}^{N} a_{2k} \;\middle|\; N \in \mathbb{N} \right\} = \sup \{ a_{2N} \mid N \in \mathbb{N} \} \underset{\text{by (vi)}}{\leq} \lambda$$

Similarly, $\bigcup_{k\in\mathbb{N}} E_{1,2k+1}$ has order type at most λ. Using Proposition 2.4.2, we conclude that E_1 has order type at most 2λ.

Using again proposition 2.4.2, point (vii) above and the properties of E_2 and E_3 mentioned in the beginning of this proof, we get that E is reverse well-ordered with order type at most $2\lambda + \omega(\gamma + 1)$. □

Corollary 5.2.5. *Let $x = \sum_{i<\nu} r_i \omega^{a_i}$ such that*

$$\exists u = \ln_n \kappa_{-\alpha} \quad \exists r \in \mathbb{R} \quad \forall a \in \operatorname{supp} x \quad \exists \eta \prec \ln u \qquad \omega^a = \partial u \exp(r \ln u + \eta)$$

Let γ be the smallest ordinal such that $\kappa_{-\gamma} \prec^K P(k_P)$ for all path $P \in \mathcal{P}_{\mathbb{L}}(\eta)$. Let λ the least ε-number greater than $\mathrm{NR}(x)$ and γ. Then $\bigcup_{\ell=0}^{+\infty} \operatorname{supp} \Phi^\ell(x)$ is reverse well-ordered with order type less at most $\omega^{\omega(2\lambda+\omega(\gamma+1)+1)}$

Proof. Just use Propositions 5.2.2, 5.2.4 and 2.4.5. □

Case $\varepsilon \succ \ln u$

We now investigate the other case following the same steps as the previous one.

Lemma 5.2.6. *Let x be a surreal number. Let P be the dominant path of x and $Q \in \mathcal{P}_{\mathbb{L}}(x)$. Then, $P(k_P) \succeq^K Q(k_Q)$. In particular, for all ordinal β, if $\kappa_{-\beta} \succeq^K P(k_P)$, then $\kappa_{-\beta} \succeq^K Q(k_Q)$.*

Proof. (i) We first claim that for all $i \in \mathbb{N}$, $P(i) \succeq Q(i)$. We prove it by induction.

- For $i = 0$, $P(0)$ is the leading term of x and $Q(0)$ is some term of x. Therefore, $P(0) \succeq Q(0)$.
- Assume $P(i) \succeq Q(i)$. $P(i+1)$ is the leading term of $\ln|P(i)|$. $P(i)$ and $Q(i)$ are both infinitely large. Then $\ln|P(i)|$ and $\ln|Q(i)|$ are both positive infinitely large. If $Q(i+1) \succ P(i+1)$ then, in particular, $\ln|Q(i)| \succ \ln|P(i)|$ what is impossible since $P(i) \succeq Q(i)$. Then $P(i+1) \succeq Q(i+1)$.

We conclude thanks to induction principle.

(ii) Take $k = \max(k_P, k_Q)$. Using (i), we have :

$$P(k_P) \asymp^K P(k) \succeq Q(k) \asymp^K Q(k_Q)$$

Hence, $P(k_P) \succeq^K Q(k_Q)$.

□

Lemma 5.2.7. *Assume $x = \omega^a = \partial u \exp \varepsilon$ with $\varepsilon \succ \ln u$ and $u = \ln_n \kappa_{-\alpha}$. Let $b \in \operatorname{supp} \Phi(\omega^a)$. Then, we have one of theses cases :*

- *there is a path $P \in \mathcal{P}(\eta)$ and $i \in \mathbb{N}$ such that*

$$\omega^b \asymp \partial u \exp \left(\varepsilon - \sum_{\substack{\beta \geq \alpha,\, m \in \mathbb{N}^* \\ \beta \mid P_0(k_{P_0}) \succ^K \kappa_{-\beta} \succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{j=0}^{+\infty} \ln \left| \frac{P(i+j)}{P_0(j)} \right| \right)$$

and $$\forall j \in [\![0 \,;\, i-1]\!] \qquad P(j) = P_0(j)$$

- *There is some $(\beta, m) <_{lex} (\alpha, n)$ such that there is some $\eta \prec \ln_m \kappa_{-\beta}$ such that*

$$\omega^b \asymp \partial(\ln_m \kappa_{-\beta}) \exp \eta$$

where $\eta = \varepsilon + \eta'$ and η' only depends on α, β, n, m and P_0, the dominant path of ε :

$$\eta' = \sum_{\substack{(\zeta,p) >_{lex} (\beta,m) \\ \zeta \mid \kappa_{-\zeta} \succeq^K P_0(k_{P_0})}} \ln_p \kappa_{-\zeta} - \sum_{(\beta,m) <_{lex} (\zeta,p) <_{lex} (\alpha,n)} \ln_p \kappa_{-\zeta} - \sum_{i=0}^{+\infty} \ln |P_0(i)|$$

or $$\eta' = \sum_{\substack{(\zeta,p) \geq_{lex} (\alpha,n) \\ \zeta \mid \kappa_{-\zeta} \succeq^K P_0(k_{P_0})}} \ln_p \kappa_{-\zeta} - \sum_{i=0}^{+\infty} \ln |P_0(i)|$$

Proof. We have
$$\Phi(\omega^a) = \left(1 - \frac{\partial\varepsilon}{s}\right)\omega^a - \partial\left(\frac{\partial u}{s}\right)\exp\varepsilon$$
Let $b \in \operatorname{supp}\Phi(\omega^a)$. Then $b \in \operatorname{supp}\left(\left(1 - \frac{\partial\varepsilon}{s}\right)\omega^a\right)$ or $b \in \operatorname{supp}\left(\partial\left(\frac{\partial u}{s}\right)\exp\varepsilon\right)$.

- First case : $b \in \operatorname{supp}\left(\left(1 - \frac{\partial\varepsilon}{s}\right)\omega^a\right)$. Then there is a path P, which is not the dominant path, such that
$$\omega^b \asymp \frac{\partial P}{s}\omega^a \asymp \frac{\exp\left(-\sum\limits_{\substack{\beta \geq \alpha, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum\limits_{i=0}^{+\infty} \ln|P(i)|\right)}{\exp\left(-\sum\limits_{\substack{\beta \geq \alpha, m \in \mathbb{N}^* \\ \beta \mid \kappa_{-\beta} \succeq^K P_0(k_{P_0})}} \ln_m \kappa_{-\beta} + \sum\limits_{i=0}^{+\infty} \ln|P_0(i)|\right)}\omega^a$$
where P_0 is the dominant path of ε. Using Lemma 5.2.6, we get
$$\omega^b \asymp \omega^a \exp\left(-\sum\limits_{\substack{\beta \geq \alpha, m \in \mathbb{N}^* \\ \beta \mid P_0(k_{P_0}) \succ^K \kappa_{-\beta} \succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum\limits_{i=0}^{+\infty} \ln\left|\frac{P(i)}{P_0(i)}\right|\right)$$

- Second case : $b \in \operatorname{supp}\left(\partial\left(\frac{\partial u}{s}\right)\exp\varepsilon\right)$. First notice that $\partial\partial u = S_u \partial u$ where
$$S_u = -\sum_{\beta<\alpha\ m\in\mathbb{N}^*} \exp\left(-\sum_{\zeta<\beta\ p\in\mathbb{N}^*} \ln_p \kappa_{-\zeta} - \sum_{p=1}^{m-1} \ln_p \kappa_{-\beta}\right) - \sum_{m=1}^{n-1} \exp\left(-\sum_{\beta<\alpha\ p\in\mathbb{N}^*} \ln_p \kappa_{-\beta} - \sum_{p=1}^{m-1} \ln_p \kappa_{-\alpha}\right)$$
Hence, if $b \in \operatorname{supp}\left(\frac{\partial\partial u}{s}\exp\varepsilon\right)$, there is some $(\beta, m) <_{lex} (\alpha, n)$ such that
$$\omega^b \asymp \omega^a \frac{\exp\left(-\sum\limits_{\zeta<\beta\ p\in\mathbb{N}^*} \ln_p \kappa_{-\zeta} - \sum\limits_{p=1}^{m-1} \ln_p \kappa_{-\beta}\right)}{\exp\left(-\sum\limits_{\substack{p \in \mathbb{N}^* \\ \zeta \mid \kappa_{-\zeta} \succeq^K P_0(k_{P_0})}} \ln_p \kappa_{-\zeta} + \sum\limits_{i=0}^{+\infty} \ln|P_0(i)|\right)}$$
Therefore,
$$\omega^b \asymp \omega^a \exp\left(\sum\limits_{\substack{(\zeta, p) \geq_{lex} (\beta, m) \\ \zeta \mid \kappa_{-\zeta} \succeq^K P_0(k_{P_0})}} \ln_p \kappa_{-\zeta} - \sum\limits_{i=0}^{+\infty} \ln|P_0(i)|\right)$$
Notice that
$$\sum\limits_{\substack{(\zeta, p) \geq_{lex} (\beta, m) \\ \zeta \mid \kappa_{-\zeta} \succeq^K P_0(k_{P_0})}} \ln_p \kappa_{-\zeta} \sim \ln_m \kappa_{-\beta} \succ P_0(0)$$
and then
$$\omega^a \asymp \partial(\ln_m \kappa_{-\beta}) \exp\left(\varepsilon + \sum\limits_{\substack{(\zeta, p) >_{lex} (\beta, m) \\ \zeta \mid \kappa_{-\zeta} \succeq^K P_0(k_{P_0})}} \ln_p \kappa_{-\zeta} - \sum\limits_{(\beta,m)<_{lex}(\zeta,p)<_{lex}(\alpha,n)} \ln_p \kappa_{-\zeta} - \sum\limits_{i=0}^{+\infty} \ln|P_0(i)|\right)$$
Since
$$\varepsilon - \sum_{i=0}^{+\infty} \ln|P(i)| \sim \varepsilon \prec \ln_m \kappa_{-\beta}$$
and
$$\sum\limits_{\substack{(\zeta, p) >_{lex} (\beta, m) \\ \zeta \mid \kappa_{-\zeta} \succeq^K P_0(k_{P_0})}} \ln_p \kappa_{-\zeta} - \sum\limits_{(\beta,m)<_{lex}(\zeta,p)<_{lex}(\alpha,n)} \ln_p \kappa_{-\zeta} \prec \ln_m \kappa_{-\beta}$$

Moreover, $$\mathrm{NR}\left(\ln\partial(\ln_m\kappa_{-\beta})+\varepsilon-\sum_{(\beta,m)<_{lex}(\zeta,p)<_{lex}(\alpha,n)}\ln_p\kappa_{-\zeta}\right)\leq\mathrm{NR}(x)$$

and using Proposition 3.9.31,

$$\mathrm{NR}\left(\sum_{\substack{(\zeta,p)>_{lex}(\beta,m)\\ \zeta\mid\kappa_{-\zeta}\succeq^K P_0(k_{P_0})}}\ln_p\kappa_{-\zeta}-\sum_{i=0}^{+\infty}\ln|P_0(i)|\right)\leq\mathrm{NR}(\partial P_0)\leq k_{P_0}(\mathrm{NR}(x)+1)+\omega(\gamma+1)$$

We then conclude that there is some $\eta\prec\ln_m\kappa_{-\beta}$ such that

$$\omega^b=\partial(\ln_m\kappa_{-\beta})\exp\eta$$

and $$\mathrm{NR}(\omega^b)\leq(k_{P_0}+1)(\mathrm{NR}(x)+1)+\omega(\gamma+1)\qquad\text{(Corollary 3.8.25)}$$

Now assume $b\in\operatorname{supp}\left(\frac{\partial s}{s^2}\omega^a\right)$. Notice that

$$\begin{aligned}\partial s&=s\left(-\sum_{\substack{m\in\mathbb{N}^*\\ \beta\mid\kappa_{-\beta}\succeq^K P_0(k_{P_0})}}\partial\ln_m\kappa_{-\beta}+\sum_{i=0}^{+\infty}\partial\ln|P_0(i)|\right)\\&=s\left(-\sum_{\substack{m\in\mathbb{N}^*\\ \beta\mid\kappa_{-\beta}\succeq^K P_0(k_{P_0})}}\exp\left(-\sum_{\zeta<\beta\ p\in\mathbb{N}^*}\ln_p\kappa_{-\zeta}-\sum_{p=1}^{m-1}\ln_p\kappa_{-\beta}\right)+\sum_{i=0}^{+\infty}\partial\ln|P_0(i)|\right)\end{aligned}$$

We then have the following sub-cases :

- ➢ There is some $m\in\mathbb{N}^*$ and some ordinal β such that $\kappa_{-\beta}\succeq^K P_0(k_{P_0})$ such that

$$\begin{aligned}\omega^b&\asymp\frac{\exp\left(-\sum_{\zeta<\beta\ p\in\mathbb{N}^*}\ln_p\kappa_{-\zeta}-\sum_{p=1}^{m-1}\ln_p\kappa_{-\beta}\right)}{\exp\left(-\sum_{\substack{p\in\mathbb{N}^*\\ \zeta\mid\kappa_{-\zeta}\succeq^K P_0(k_{P_0})}}\ln_p\kappa_{-\zeta}+\sum_{i=0}^{+\infty}\ln|P_0(i)|\right)}\omega^a\\&\asymp\partial(\ln_m\kappa_{-\beta})\exp\left(\varepsilon+\sum_{\substack{(\zeta,p)\geq_{lex}(\alpha,n)\\ \zeta\mid\kappa_{-\zeta}\succeq^K P_0(k_{P_0})}}\ln_p\kappa_{-\zeta}-\sum_{i=0}^{+\infty}\ln|P_0(i)|\right)\end{aligned}$$

with $$\varepsilon-\sum_{\substack{\zeta\geq\alpha\ p\in\mathbb{N}^*\\ \zeta\mid\kappa_{-\zeta}\succeq^K P_0(k_{P_0})}}\ln_p\kappa_{-\zeta}-\sum_{i=0}^{+\infty}\ln|P_0(i)|\sim\varepsilon\prec\ln_m\kappa_{-\beta}$$

We then conclude that there is some $\eta\prec\ln_m\kappa_{-\beta}$ such that

$$\omega^b=\partial(\ln_m\kappa_{-\beta})\exp\eta$$

and $$\mathrm{NR}(\omega^b)\leq(k_{P_0}+1)(\mathrm{NR}(x)+1)+\omega(\gamma+1)\qquad\text{(Corollary 3.8.25)}$$

- ➢ There is some path $P\in\mathcal{P}_{\mathbb{L}}(\varepsilon)$ and some $i\geq 1$ such that for all $j<i$, $P(j)=P_0(j)$ and

$$\omega^b\asymp\frac{\exp\left(-\sum_{\substack{p\in\mathbb{N}^*\\ \zeta\mid\kappa_{-\zeta}\succeq^K P(k_P)}}\ln_p\kappa_{-\zeta}+\sum_{j=i}^{+\infty}\ln|P(j)|\right)}{\exp\left(-\sum_{\substack{p\in\mathbb{N}^*\\ \zeta\mid\kappa_{-\zeta}\succeq^K P_0(k_{P_0})}}\ln_p\kappa_{-\zeta}+\sum_{j=0}^{+\infty}\ln|P_0(j)|\right)}\omega^a$$

As in the first case, we get

$$\omega^b \asymp \omega^a \exp\left(-\sum_{\substack{\beta \geq \alpha, m \in \mathbb{N}^* \\ \beta \mid P_0(k_{P_0}) \succ^K \kappa_{-\beta} \succeq^K P(k_P)}} \ln_m \kappa_{-\beta} + \sum_{j=0}^{+\infty} \ln\left|\frac{P(i+j)}{P_0(j)}\right|\right)$$

□

Proposition 5.2.8. *Assume $x = \omega^a = \partial u \exp \varepsilon$ with $\varepsilon \succ \ln u$ and $u = \ln_n \kappa_{-\alpha}$. Let P_0 be the dominant path of ε. We denote for $P_1, \ldots, P_{k+k'} \in \mathcal{P}_{\mathbb{L}}(\varepsilon)$, $i_1, \ldots, i_{k+k'} \in \mathbb{N}^*$ and $(\beta, m) \leq_{lex} (\alpha, n)$,*

$$\begin{aligned} e^{(\beta,m)}\begin{pmatrix} P_1,\ldots,P_k \\ P_{k+1},\ldots,P_{k+k'} \\ i_1,\ldots,i_{k+k'}\end{pmatrix} &= -k\sum_{i=0}^{\infty}\ln|P_0(i)| - \sum_{j=1}^{k} \sum_{\substack{\gamma \geq \alpha, \ell \in \mathbb{N}^* \\ \gamma \mid P_0(k_{P_0}) \succ^K \kappa_{-\gamma} \succeq^K P_j(k_{P_j})}} \ln_\ell \kappa_{-\gamma} + \sum_{j=1}^{k}\sum_{i=i_j}^{\infty} \ln|P_j(i)| \\ &\quad -k'\sum_{\ell=m+2}^{+\infty} \ln_\ell \kappa_{-\beta} - \sum_{j=k+1}^{k+k'} \sum_{\substack{\gamma > \beta, \ell \in \mathbb{N}^* \\ \gamma \mid \kappa_{-\gamma} \succeq^K P_j(k_{P_j})}} \ln_\ell \kappa_{-\gamma} + \sum_{j=k+1}^{k+k'}\sum_{i=i_j}^{+\infty} \ln|P_j(i)| \end{aligned}$$

Let

$$E_{1,k,k'}^{(\beta,m)} = \left\{ e^{(\beta,m)}\begin{pmatrix} P_1,\ldots,P_k \\ P_{k+1},\ldots,P_{k+k'} \\ i_1,\ldots,i_{k+k'}\end{pmatrix} \;\middle|\; \begin{array}{c} P_1,\ldots,P_k \in \mathcal{P}_{\mathbb{L}}(\varepsilon)\setminus\{P_0\} \quad P_{k+1},\ldots,P_{k+k'} \in \mathcal{P}_{\mathbb{L}}(\varepsilon) \\ i_1,\ldots,i_k \in \mathbb{N} \quad i_{k+1},\ldots,i_{k+k'} \in \mathbb{N}^* \\ \forall j \in [\![1 \,;\, k+k']\!] \; \exists j' \in [\![0 \,;\, j-1]\!] \; \forall i \in [\![0 \,;\, i_j - 1]\!] \quad P_{j'}(i) = P_j(i) \\ \forall j \in [\![k+1 \,;\, k+k']\!] \quad \operatorname{supp} P_j(i_j) \subseteq \operatorname{supp} e^{(\beta,m)}\begin{pmatrix} P_1,\ldots,P_k \\ P_{k+1},\ldots,P_j \\ i_1,\ldots,i_j\end{pmatrix} \end{array}\right\}$$

$$E_1^{(\beta,m)} = \begin{cases} \bigcup_{k\in\mathbb{N},\, k'\in\mathbb{N}^*} E_{1,k,k'}^{(\beta,m)} & (\beta,m) \neq (\alpha,n) \\ \bigcup_{k\in\mathbb{N}} E_{1,k,0}^{(\beta,m)} & (\beta,m) = (\alpha,n) \end{cases}$$

$$E_2^{(\beta,m)} = \begin{cases} \left\{ -\sum_{\ell=m+2}^{+\infty} \ln_\ell \kappa_{-\beta} - \sum_{\gamma'<\gamma,\, \ell\in\mathbb{N}^*} \ln_\ell \kappa_{-\gamma'} - \sum_{\ell=1}^{p} \ln_\ell \kappa_{-\gamma} \;\middle|\; \begin{array}{c} \gamma > \beta \\ \exists P \in \mathcal{P}_{\mathbb{L}}(\varepsilon) \quad \kappa_{-\gamma} \succeq^K P(k_P) \\ p \in \mathbb{N} \end{array} \right\} & (\beta,m) \neq (\alpha,n) \\ \left\{ -\sum_{j=0}^{\infty} \ln|P_0(j)| - \sum_{\gamma>\zeta>\alpha,\, \ell\in\mathbb{N}^*} \ln_\ell \kappa_{-\zeta} - \sum_{\ell=1}^{p} \ln_\ell \kappa_{-\gamma} \;\middle|\; \begin{array}{c} \gamma > \alpha \quad P_0(k_{P_0}) \succ^K \kappa_{-\gamma} \\ \exists P \in \mathcal{P}_{\mathbb{L}}(\varepsilon) \quad \kappa_{-\gamma} \succeq^K P(k_P) \\ p \in \mathbb{N} \end{array} \right\} & (\beta,m) = (\alpha,n) \end{cases}$$

$$E_3^{(\beta,m)} = \begin{cases} \left\{ -\sum_{\ell=m+2}^{p} \ln_\ell \kappa_{-\beta} \;\middle|\; p \geq m+2 \right\} & (\beta,m) \neq (\alpha,n) \\ \varnothing & (\beta,m) = (\alpha,n) \end{cases}$$

$$E^{(\beta,m)} = E_1^{(\beta,m)} \cup E_2^{(\beta,m)} \cup E_3^{(\beta,m)}$$

and $\langle E^{(\beta,m)} \rangle$ be the monoid it generates. Finally, let

$$H^{(\beta,m)} = \begin{cases} \left\{ \begin{array}{l} \sum_{\substack{(\zeta,p) >_{lex} (\beta,m) \\ \zeta \mid \kappa_{-\zeta} \succeq^K P_0(k_{P_0})}} \ln_p \kappa_{-\zeta} - \sum_{(\beta,m)<_{lex}(\zeta,p)<_{lex}(\alpha,n)} \ln_p \kappa_{-\zeta} - \sum_{i=0}^{+\infty} \ln|P_0(i)|, \\ \sum_{\substack{(\zeta,p) \geq_{lex} (\alpha,n) \\ \zeta \mid \kappa_{-\zeta} \succeq^K P_0(k_{P_0})}} \ln_p \kappa_{-\zeta} - \sum_{i=0}^{+\infty} \ln|P_0(i)| \end{array} \right\} & (\beta,m) \neq (\alpha,n) \\ \{0\} & (\beta,m) = (\alpha,n) \end{cases}$$

Let $b \in \bigcup_{q=0}^{+\infty} \operatorname{supp} \Phi^q(\omega^a)$. Then, there are $\eta \in H^{(\beta,m)}$ and $y \in \langle E^{(\beta,m)} \rangle$ such that

$$\omega^b \asymp \partial(\ln_m \kappa_{-\beta}) \exp(\varepsilon + \eta + y)$$

Proof. We prove it by induction on q.

- If $b \in \operatorname{supp} \Phi^0(\omega^a)$, then $b = a$ and $y = 0$ with $(\beta, m) = (\alpha, n)$ and $\eta = 0$ works.
- Assume the property for some $q \in \mathbb{N}$. Let $b \in \operatorname{supp} \Phi^{q+1}(\omega^b)$. Then there is $c \in \operatorname{supp} \Phi^q(\omega^a)$ such that $b \in \operatorname{supp} \Phi(\omega^c)$. Apply the induction hypothesis on c. Take (β, m), $\eta \in H^{(\beta,m)}$ and $y \in \langle E^{(\beta,m)} \rangle$ such that

$$\omega^c \asymp \partial(\ln_m \beta) \exp(\varepsilon + \eta + y)$$

➢ If $(\beta,m) <_{lex} (\alpha,n)$, then $y,\varepsilon \prec \ln_{n+1}\kappa_{-\beta}$. Hence, using Lemma 5.2.1, we get that there is $P\in\mathcal{P}_{\mathbb{L}}(\varepsilon+\eta+y)$ such that

$$\omega^b \asymp \partial(\ln_m\kappa_{-\beta})\exp\left(\varepsilon+\eta+y-\sum_{\ell=m+2}^{+\infty}\ln_\ell\kappa_{-\beta}-\sum_{\substack{\gamma>\beta,\ell\in\mathbb{N}^*\\ \gamma\,|\,\kappa_{-\gamma}\succeq^K P(k_P)}}\ln_\ell\kappa_{-\gamma}+\sum_{i=0}^{+\infty}\ln|P(i)|\right)$$

If $P(0)$ a term of ε, up to some real factor, then there is a real number s and some $e\in E_{1,0,1}^{(\beta,m)}$ such that

$$\exp\left(-\sum_{\ell=m+2}^{+\infty}\ln_\ell\kappa_{-\beta}-\sum_{\substack{\gamma>\beta,\ell\in\mathbb{N}^*\\ \gamma\,|\,\kappa_{-\gamma}\succeq^K P(k_P)}}\ln_\ell\kappa_{-\gamma}+\sum_{i=0}^{+\infty}\ln|P(i)|\right)=s\exp e$$

Then $y+e\in\left\langle E^{(\beta,m)}\right\rangle$ and $\omega^b\asymp\partial(\ln_m\kappa_{-\beta})\exp(\varepsilon+y+e)$. If not, then $P(0)$ is a term of $\eta+y$. Hence, we have the following cases :

∵ $P(0)=s\ln_p\kappa_{-\beta}$ for some $s\in\mathbb{R}^*_-$ and $p\geq m+2$. Then,

$$-\sum_{\ell=m+2}^{+\infty}\ln_\ell\kappa_{-\beta}-\sum_{\substack{\gamma>\beta,\ell\in\mathbb{N}^*\\ \gamma\,|\,\kappa_{-\gamma}\succeq^K P(k_P)}}\ln_\ell\kappa_{-\gamma}+\sum_{i=0}^{+\infty}\ln|P(i)|=\ln|s|-\sum_{\ell=m+2}^{p}\ln_\ell\kappa_{-\beta}\in\ln|s|+E_3^{(\beta,m)}$$

Then $$y-\sum_{\ell=m+2}^{+\infty}\ln_\ell\kappa_{-\beta}-\sum_{\substack{\gamma>\beta,\ell\in\mathbb{N}^*\\ \gamma\,|\,\kappa_{-\gamma}\succeq^K P(k_P)}}\ln_\ell\kappa_{-\gamma}+\sum_{i=0}^{+\infty}\ln|P(i)|\in\mathbb{R}+\left\langle E^{(\beta,m)}\right\rangle$$

∵ $P(0)=s\ln_p\kappa_{-\gamma}$ with $\gamma>\beta$ and $p\in\mathbb{N}^*$ such that there is some path $Q\in\mathcal{P}_{\mathbb{L}}(\varepsilon)$ such that $\kappa_{-\beta}\succeq^K Q(k_Q)$. Then

$$-\sum_{\ell=m+2}^{+\infty}\ln_\ell\kappa_{-\beta}-\sum_{\substack{\gamma>\beta,\ell\in\mathbb{N}^*\\ \gamma\,|\,\kappa_{-\gamma}\succeq^K P(k_P)}}\ln_\ell\kappa_{-\gamma}+\sum_{i=0}^{+\infty}\ln|P(i)|\in\ln|s|+E_2^{(\beta,m)}$$

Then $$y-\sum_{\ell=m+2}^{+\infty}\ln_\ell\kappa_{-\beta}-\sum_{\substack{\gamma>\beta,\ell\in\mathbb{N}^*\\ \gamma\,|\,\kappa_{-\gamma}\succeq^K P(k_P)}}\ln_\ell\kappa_{-\gamma}+\sum_{i=0}^{+\infty}\ln|P(i)|\in\mathbb{R}+\left\langle E^{(\beta,m)}\right\rangle$$

∵ There are some paths $P_1,\dots,P_{k+k'}\in\mathcal{P}_{\mathbb{L}}(\varepsilon)$ and some integers $i_1,\dots,i_{k+k'}$ such that

$$e^{(\beta,m)}\begin{pmatrix}P_1,\dots,P_k\\ P_{k+1},\dots,P_{k+k'}\\ i_1,\dots,i_{k+k'}\end{pmatrix}\in E_{1,k,k'}^{(\beta,m)}$$

and $$\exists y'\in\langle E\rangle\qquad y=y'+e^{(\beta,m)}\begin{pmatrix}P_1,\dots,P_k\\ P_{k+1},\dots,P_{k+k'}\\ i_1,\dots,i_{k+k'}\end{pmatrix}$$

and such that $P(0)\in\mathbb{R}z$ for some z a term of some $\ln|P_j(i_{k+k'+1}{}')|$ with $j\in[\![0\,;\,k+k']\!]$ and $i_{k+k'+1}{}'\geq i_j$. Let $P_{k+k'+1}$ be the following path :

$$P_{k+k'+1}(i)=\begin{cases}P_j(i) & i\leq i_{k+1}{}'\\ z & i=i_{k+1}{}'+1\\ P(i-i_{k+1}{}'-1) & i>i_{k+1}{}'+1\end{cases}$$

Then, $P_{k+k'+1}\in\mathcal{P}(\varepsilon)$. Moreover, $\partial P_{k+k'+1}=\underbrace{P_j(0)\cdots P_j(i_{k+1}{}')}_{\neq 0}\underbrace{\partial P}_{\neq 0}$. Then $P_{k+k'+1}\in\mathcal{P}_{\mathbb{L}}(\varepsilon)$. Note also that for all β,

$$\kappa_{-\beta}\succeq^K P_{k+k'+1}(k_{P_{k+k'+1}})\iff\kappa_{-\beta}\succeq^K P(k_P)$$

Finally,

$$\begin{aligned}&-\sum_{\ell=m+2}^{+\infty}\ln_\ell\kappa_{-\beta}-\sum_{\substack{\gamma>\beta,\ell\in\mathbb{N}^*\\ \gamma\,|\,\kappa_{-\gamma}\succeq^K P(k_P)}}\ln_\ell\kappa_{-\gamma}+\sum_{i=0}^{+\infty}\ln|P(i)|\\ &=-\sum_{\ell=m+2}^{+\infty}\ln_\ell\kappa_{-\beta}-\sum_{\substack{\gamma>\beta,\ell\in\mathbb{N}^*\\ \gamma\,|\,\kappa_{-\gamma}\succeq^K P_{k+k'+1}(k_{P_{k+k'+1}})}}\ln_\ell\kappa_{-\gamma}+\sum_{i=i_{k+1}{}'+1}^{+\infty}\ln|P_{k+k'+1}(i)|+\underbrace{\ln\left|\frac{P(0)}{z}\right|}_{\in\mathbb{R}^*_+}\end{aligned}$$

From that we derive that

$$y-\sum_{\ell=m+2}^{+\infty}\ln_\ell \kappa_{-\beta}-\sum_{\substack{\gamma>\beta,\,\ell\in\mathbb{N}^*\\ \gamma\,|\,\kappa_{-\gamma}\succeq^K P(k_P)}}\ln_\ell \kappa_{-\gamma}+\sum_{i=0}^{+\infty}\ln|P(i)|$$
$$=y'+e^{(\beta,m)}\begin{pmatrix}P_1,\dots,P_k\\ P_{k+1},\dots,P_{k+k'+1}\\ i_1,\dots,i_{k+k'+1}\end{pmatrix}+\ln\left|\frac{P(0)}{z}\right|\in\mathbb{R}+\left\langle E^{(\beta,m)}\right\rangle$$

where $i_{k+k'+1}={i_{k+k'+1}}'+1$ and $P_{k+k'+1}(i_{k+k'})=z$ has indeed its support (which is reduced to a singleton) included in the one of $e^{(\beta,m)}\begin{pmatrix}P_1,\dots,P_k\\ P_{k+k'},\dots,P_{k+k'}\\ i_1,\dots,i_{k+k'}\end{pmatrix}$.

Then there is a real number s, and $e\in\left\langle E^{(\beta,m)}\right\rangle$ such that

$$\omega^b\asymp\partial(\ln_m\beta)\exp(\varepsilon+\eta+e+s)\asymp\partial(\ln_m\beta)\exp(\varepsilon+\eta+e)$$

Then we get the property at rank $q+1$.

➢ If $(\beta,m)=(\alpha,n)$, we have $\eta=0$ and write

$$y=y'+e^{(\alpha,n)}\begin{pmatrix}P_1,\dots,P_k\\ \varnothing\\ i_1,\dots,i_{k+k'}\end{pmatrix}$$

with, $y'\in\left\langle E^{(\beta,m)}\right\rangle$ and, $k,k'\in\mathbb{N}$. Using Lemma 5.2.7, we have

∵ First case : $$\omega^b\asymp\partial u\exp(\varepsilon+y+e)$$

where $$e=-\sum_{\substack{\gamma\geq\alpha,\,\ell\in\mathbb{N}^*\\ \gamma\,|\,P_0(k_{P_0})\succ^K\kappa_{-\gamma}\succeq^K P(k_P)}}\ln_\ell\kappa_{-\gamma}+\sum_{j=0}^{+\infty}\ln\left|\frac{P(i+j)}{P_0(j)}\right|$$

for some path $P\in\mathcal{P}_{\mathbb{L}}(\varepsilon+y)$ and some $i\in\mathbb{N}$ such that

$$\forall j\in[\![0\,;\,i-1]\!]\qquad P(j)=P_0(j)$$

Indeed, $y\in\left\langle E^{(\alpha,n)}\right\rangle$. In particular, $y\prec\varepsilon$ and then $\varepsilon+y\sim\varepsilon$ so that P_0 is also the dominant path of $\varepsilon+y$.

- If $P(0)$ is, up to a real factor, a term of ε, then we get that there is some path $Q\in\mathcal{P}_{\mathbb{L}}(\varepsilon)$ and a real number s such that

$$y+e=y'+e^{(\beta,m)}\begin{pmatrix}P_1,\dots,P_k,Q\\ \varnothing\\ i_1,\dots,i_k,i\end{pmatrix}+s$$

Since $y\prec\varepsilon$, and $P\neq P_0$, we also have $Q\neq P_0$. Then $y+e\in\left\langle E^{(\beta,m)}\right\rangle+E^{(\beta,m)}_{1,k+1,k'}+s$. Let

$$y''=y+e-s\in\left\langle E^{(\beta,m)}\right\rangle$$

then, $$\omega^b\asymp\partial u\exp(\varepsilon+y'')$$

In particular, $y''\prec\varepsilon$.

- If $P(0)$ is a term of y, and more precisely if it can be written as $P(0)=s\ln_p\kappa_{-\gamma}$ for $s\in\mathbb{R}$, $p\in\mathbb{N}$ and $\gamma\geq\alpha$ such that

$$P_0(k_{P_0})\succ^K\kappa_{-\gamma}\succeq^K Q(k_Q)$$

for some path $Q\in\mathcal{P}_{\mathbb{L}}(\varepsilon)\setminus\{P_0\}$. Then,

$$e=-\sum_{j=0}^{\infty}\ln|P_0(j)|-\sum_{\gamma>\zeta>\alpha,\ \ell\in\mathbb{N}^*}\ln_\ell\kappa_{-\zeta}-\sum_{\ell=1}^{p+i}\ln_\ell\kappa_{-\gamma}+\mathbb{1}_{i=0}\ln|s|\in E_2^{(\beta,m)}+\mathbb{R}$$

Then $y+e-\ln|s|\in\left\langle E^{(\beta,m)}\right\rangle$ and since $e\prec\varepsilon$, $y+e-s\prec\varepsilon$ and

$$\omega^b\asymp\partial u\exp(\varepsilon+y+e-\ln|s|)$$

- If $P(0)$ is a term of y, and more precisely if it can be written as $P(0)=s\ln|P_\ell(j)|$ for some $s\in\mathbb{R}$ and some $\ell\in[\![0\,;\,k+k']\!]$ (actually it is true if we have chosen well the y' in the beginning, but up to a renaming, it is true). Consider the following path

$$Q(p)=\begin{cases}P_\ell(p) & p\leq j\\ P(p-j) & p>j\end{cases}$$

We have $Q\in\mathcal{P}_{\mathbb{L}}(\varepsilon)$ and

$$y+e=y'+e^{(\beta,m)}\begin{pmatrix}P_1,\dots,P_k,Q\\ \varnothing\\ i_1,\dots,i_k,j\end{pmatrix}+\ln|s|$$

Then $y+e-\ln|s|\in\left\langle E^{(\beta,m)}\right\rangle$ and since $e\prec\varepsilon$, $y+e-s\prec\varepsilon$ and

$$\omega^b\asymp\partial u\exp(\varepsilon+y+e-\ln|s|)$$

This concludes the first case.

∵ <u>Second case</u> : There are $(\beta', m') <_{lex} (\alpha, n)$ and $\eta' \in H^{(\beta,m)}$ such that $\omega^b \asymp \partial(\ln_{m'}\kappa_{-\beta'})\exp(\varepsilon + \eta' + y)$. This immediately conclude the second case.

We then have the property at rank $q+1$.

Thanks to the induction principle, we conclude that the property holds for any $q \in \mathbb{N}$. □

Corollary 5.2.9. *Let x be a surreal number such that*

$$\exists u = \ln_n \kappa_{-\alpha} \quad \exists r_0 \in \mathbb{R}^* \quad \exists a_0 \in \mathbf{No} \quad \forall a \in \operatorname{supp} x \quad \exists \varepsilon \sim r\omega^{a_0} \succ \ln u \qquad \omega^a \asymp \partial u \exp \varepsilon$$

Let
$$\mathcal{P}_0(x) = \left\{ P \in \mathcal{P}_{\mathbb{L}}(x) \;\middle|\; \begin{array}{c} P(1) = r\omega^{a_0} \\ \forall i \geq 1 \quad P(i+1) \sim \ln|P(i)| \end{array} \right\}$$

It is the set of all the possible dominant paths of the epsilon to which we add the corresponding term of x at the beginning. We denote for $P_0 \in \mathcal{P}_0(x)$, $P_1, \dots, P_{k+k'} \in \mathcal{P}_{\mathbb{L}}(x)$, $i_1, \dots, i_{k+k'} \in \mathbb{N}^*$ and $(\beta, m) \leq_{lex} (\alpha, n)$,

$$\begin{aligned} e^{(\beta,m)}\begin{pmatrix} P_0;P_1,\dots,P_k \\ P_{k+1},\dots,P_{k+k'} \\ i_1,\dots,i_{k+k'} \end{pmatrix} = & -k\sum_{i=1}^{\infty}\ln|P_0(i)| - \sum_{j=1}^{k} \sum_{\substack{\gamma \geq \alpha,\, \ell \in \mathbb{N}^* \\ \gamma \mid P_0(k_{P_0}) \succ^K \kappa_{-\gamma} \succeq^K P_j(k_{P_j})}} \ln_\ell \kappa_{-\gamma} + \sum_{j=1}^{k}\sum_{i=i_j}^{\infty}\ln|P_j(i)| \\ & -k'\sum_{\ell=m+2}^{+\infty}\ln_\ell\kappa_{-\beta} - \sum_{j=k+1}^{k+k'} \sum_{\substack{\gamma > \beta,\, \ell \in \mathbb{N}^* \\ \gamma \mid \kappa_{-\gamma} \succeq^K P_j(k_{P_j})}} \ln_\ell \kappa_{-\gamma} + \sum_{j=k+1}^{k+k'}\sum_{i=i_j}^{+\infty}\ln|P_j(i)| \end{aligned}$$

Let

$$E_{1,k,k'}^{(\beta,m)} = \left\{ e^{(\beta,m)}\begin{pmatrix} P_0;P_1,\dots,P_k \\ P_{k+1},\dots,P_{k+k'} \\ i_1,\dots,i_{k+k'} \end{pmatrix} \;\middle|\; \begin{array}{c} P_0 \in \mathcal{P}_0(x) \quad P_1, \dots P_k \in \mathcal{P}(x) \setminus \{P_0\} \quad P_{k+1}, \dots, P_{k+k'} \in \mathcal{P}_{\mathbb{L}}(x) \\ i_1, \dots, i_k \in \mathbb{N}^* \quad i_{k+1}, \dots, i_{k+k'} \in \mathbb{N} \setminus \{0,1\} \\ \forall j \in [\![1 \,;\, k+k']\!] \; \exists j' \in [\![0 \,;\, j-1]\!] \; \forall i \in [\![0 \,;\, i_j - 1]\!] \quad P_{j'}(i) = P_j(i) \\ \forall j \in [\![k+1 \,;\, k+k']\!] \quad \operatorname{supp} P_j(i_j) \subseteq \operatorname{supp} e^{(\beta,m)}\begin{pmatrix} P_0;P_1,\dots,P_k \\ P_{k+1},\dots,P_j \\ i_1,\dots,i_j \end{pmatrix} \end{array} \right\}$$

$$E_1^{(\beta,m)} = \begin{cases} \displaystyle\bigcup_{k\in\mathbb{N},\, k'\in\mathbb{N}^*} E_{1,k,k'}^{(\beta,m)} & (\beta,m) \neq (\alpha,n) \\ \displaystyle\bigcup_{k\in\mathbb{N}} E_{1,k,0}^{(\beta,m)} & (\beta,m) = (\alpha,n) \end{cases}$$

$$E_2^{(\beta,m)} = \begin{cases} \left\{ -\displaystyle\sum_{\ell=m+2}^{+\infty}\ln_\ell\kappa_{-\beta} - \sum_{\gamma'<\gamma,\, \ell\in\mathbb{N}^*}\ln_\ell\kappa_{-\gamma'} - \sum_{\ell=1}^{p}\ln_\ell\kappa_{-\gamma} \;\middle|\; \begin{array}{c} \gamma > \beta \\ \exists P \in \mathcal{P}_{\mathbb{L}}(x) \quad \kappa_{-\gamma} \succeq^K P(k_P) \\ p \in \mathbb{N} \end{array} \right\} & (\beta,m) \neq (\alpha,n) \\ \left\{ -\displaystyle\sum_{j=1}^{\infty}\ln|P_0(j)| - \sum_{\gamma>\zeta>\alpha,\, \ell\in\mathbb{N}^*}\ln_\ell\kappa_{-\zeta} - \sum_{\ell=1}^{p}\ln_\ell\kappa_{-\gamma} \;\middle|\; \begin{array}{c} \gamma > \alpha \quad P_0 \in \mathcal{P}_0(x) \\ P_0(k_{P_0}) \succ^K \kappa_{-\gamma} \\ \exists P \in \mathcal{P}_{\mathbb{L}}(x) \quad \kappa_{-\gamma} \succeq^K P(k_P) \\ p \in \mathbb{N} \end{array} \right\} & (\beta,m) = (\alpha,n) \end{cases}$$

$$E_3^{(\beta,m)} = \begin{cases} \left\{ -\displaystyle\sum_{\ell=m+2}^{p}\ln_\ell\kappa_{-\beta} \;\middle|\; p \geq m+2 \right\} & (\beta,m) \neq (\alpha,n) \\ \varnothing & (\beta,m) = (\alpha,n) \end{cases}$$

$E^{(\beta,m)} = E_1^{(\beta,m)} \cup E_2^{(\beta,m)} \cup E_3^{(\beta,m)}$

and $\left\langle E^{(\beta,m)} \right\rangle$ be the monoid it generates. Finally, let

$$H^{(\beta,m)} = \begin{cases} \left\{ \displaystyle\sum_{\substack{(\zeta,p) >_{lex} (\beta,m) \\ \zeta \mid \kappa_{-\zeta} \succeq^K P_0(k_{P_0})}} \ln_p\kappa_{-\zeta} - \sum_{(\beta,m)<_{lex}(\zeta,p)<_{lex}(\alpha,n)} \ln_p\kappa_{-\zeta} - \sum_{i=0}^{+\infty}\ln|P_0(i)| \;\middle|\; P_0 \in \mathcal{P}_0(x) \right\} & \\ \qquad \bigcup \left\{ \displaystyle\sum_{\substack{(\zeta,p) \geq_{lex} (\alpha,n) \\ \zeta \mid \kappa_{-\zeta} \succeq^K P_0(k_{P_0})}} \ln_p\kappa_{-\zeta} - \sum_{i=0}^{+\infty}\ln|P_0(i)| \;\middle|\; P_0 \in \mathcal{P}_0(x) \right\} & (\beta,m) \neq (\alpha,n) \\ \{-\ln|P_0(x)| \mid P_0 \in \mathcal{P}_0(x)\} & (\beta,m) = (\alpha,n) \end{cases}$$

Let $b \in \displaystyle\bigcup_{q=0}^{+\infty} \operatorname{supp}\Phi^q(x)$. Then, there are $\eta \in H^{(\beta,m)}$ and $y \in \left\langle E^{(\beta,m)} \right\rangle$ such that

$$\omega^b \asymp \frac{\partial \ln_m \kappa_{-\beta}}{\partial u}\exp(\eta + y)$$

Proof. Since Φ is strongly linear, we just need to apply Proposition 5.2.8 to each term of x. Each path of $\mathcal{P}_0(x)$ involved is shifted one rank. In $H^{(\beta,m)}$ we the add $\ln|P_0(0)|$ compare to Proposition 5.2.8. Then $\exp(\eta)$ gives also $|\partial u \exp\varepsilon|$. We just remove it so that it does not appear twice. □

Proposition 5.2.10. *Let x be a surreal number such that*

$$\exists u = \ln_n \kappa_{-\alpha} \quad \exists r_0 \in \mathbb{R}^* \quad \exists a_0 \in \mathbf{No} \quad \forall a \in \operatorname{supp} x \quad \exists \varepsilon \sim r\omega^{a_0} \succ \ln u \qquad \omega^a \asymp \partial u \exp\varepsilon$$

Let

$$\mathcal{P}_0(x) = \left\{ P \in \mathcal{P}_{\mathbb{L}}(x) \;\middle|\; \begin{array}{c} P(1) = r\omega^{a_0} \\ \forall i \geq 1 \quad P(i+1) \sim \ln|P(i)| \end{array} \right\}$$

Consider $E_1^{(\beta,m)}$, $E_2^{(\beta,m)}$ and $E_3^{(\beta,m)}$ as defined in Corollary 5.2.9. Let ξ be the smallest ordinal such that $\kappa_{-\xi} \prec^K P(k_P)$ for all path $P \in \mathcal{P}_{\mathbb{L}}(x)$. Let λ the least ε-number greater than $\mathrm{NR}(x)$ and ξ. Then $E^{(\beta,m)} = E_1^{(\beta,m)} \cup E_2^{(\beta,m)} \cup E_3^{(\beta,m)}$ is reverse well-ordered with order type at most $2\lambda + \omega(\xi+1)$.

Proof. First notice that $E_3^{(\beta,m)}$ is reverse well-ordered with order type at most ω. $E_2^{(\beta,m)}$ is also reverse well-ordered with order at most $\omega \otimes \xi$. We then focus on $E_1^{(\beta,m)}$. For the moment, we will assume $(\beta,m) <_{lex} (\alpha,n)$.

(i) We first claim that for all $i \geq 3$ and all path $P \in \mathcal{P}(x)$, $P(i) \prec P(2) \preceq \ln_{m+2}\kappa_{-\beta}$. It is indeed the same proof as the point (i) of the proof of Proposition 5.2.4.

(ii) We claim that for all path $P \in \mathcal{P}(x)$, if $P(2) \asymp \ln_{m+2}\kappa_{-\beta}$, then, if r is the real number such that $P(2) \sim r\ln_{m+2}\kappa_{-\beta}$, we have $0 < r \leq 1$. It is indeed the same proof as the point (ii) of the proof of Proposition 5.2.4.

(iii) For all j and $i \geq 2$, $\ln|P_j(i)| \preceq \ln_{m+3}\kappa_{-\beta} \prec \ln_{m+2}\kappa_{-\beta}$. Indeed, using (i), we know that $P_j(i) \preceq \ln_{m+2}\kappa_{-\beta}$. Then, there is a natural number $m \geq 1$ such that $|P_j(i)| \leq m\ln_{m+2}\kappa_{-\beta}$. Using the fact that ln is increasing,

$$\ln|P_j(i)| \leq \ln_{m+3}\kappa_{-\beta} + \ln m \preceq \ln_{m+3}\kappa_{-\beta} \prec \ln_{m+2}\kappa_{-\beta}$$

(iv) We now claim that $\bigcup_{k\in\mathbb{N}} E_{1,k,k'}^{(\beta,m)} > \bigcup_{k\in\mathbb{N}} E_{1,k,k'+2}^{(\beta,m)}$. Indeed, using (ii) and (iii) if $e_1 \in \bigcup_{k\in\mathbb{N}} E_{1,k,k'}^{(\beta,m)}$, then there is $s \in [\,-(k+1)\,;-k\,]$ such that $e_1 \sim s\ln_{m+2}\kappa_{-\beta}$. Similarly, for $e_2 \in \bigcup_{k\in\mathbb{N}} E_{1,k,k'+2}^{(\beta,m)}$ there is a real number $s' \in [\,-(k+3)\,;-(k+2)\,]$ such that $e_2 \sim s'\ln_{m+2}\kappa_{-\beta}$.

(v) We define the following sequence :

- $a_0 = 1$
- $a_{k+1} = \omega^{\omega^{\omega(\omega(\mathrm{NR}(x)+\xi+1)a_k+1)}}$

We show that $E_{1,k,0}^{(\beta,m)}$ is reverse well-ordered with order type less than a_k. We also claim that the equivalence classes of $E_{1,k,0}^{(\beta,m)}/\asymp$ are finite and that

$$\mathrm{NR}\left(\sum_{t\in E_{1,k,0}^{(\beta,m)}} \exp t\right) \leq \omega(\mathrm{NR}(x)+\xi+1)a_k$$

We show it by induction on $k \in \mathbb{N}$.

- For $k=0$, $E_{1,0,0}^{(\beta,m)} = \{0\}$. Then it is reverse well-ordered with order type 1. We also have

$$\mathrm{NR}\left(\sum_{t\in E_{1,0,0}^{(\beta,m)}} \exp t\right) = \mathrm{NR}(1) = 1 \leq \omega(\mathrm{NR}(x)+\xi+1)$$

- Assume the property for some $k \in \mathbb{N}$. Let $t \in E_{1,k+1,0}^{(\beta,m)}$. Let $(P_0,0),(P_1,i_1),\ldots,(P_{k+1},i_{k+1})$ minimal for the order $(<_{lex},<)_{lex}$ such that

$$t = e^{(\beta,m)}\begin{pmatrix} P_0; P_1,\ldots,P_{k+1} \\ \varnothing \\ i_1,\ldots,i_{k+1} \end{pmatrix}$$

Then,

$$t = e^{(\beta,m)}\begin{pmatrix} P_0; P_1\ldots,P_k \\ \varnothing \\ i_1,\ldots,i_k \end{pmatrix} - \sum_{i=1}^{+\infty}\ln|P_0(i)| - \sum_{\substack{\gamma \geq \alpha,\, m\in\mathbb{N}^* \\ \gamma \mid P_0(k_{P_0}) \succ^K \kappa_{-\gamma} \succeq^K P_{k+1}(k_{P_{k+1}})}} \ln_m\kappa_{-\gamma} + \sum_{i=i_{k+1}}^{+\infty}\ln|P_{k+1}(i)|$$

Write $s = e^{(\beta,m)}\begin{pmatrix} P_0; P_1, \dots, P_k \\ \varnothing \\ i_1, \dots, i_k \end{pmatrix}$ and consider the following path :

$$\begin{cases} R(0) = \exp s \\ R(i) = P_{k+1}(i - 1 + i_{k+1}) \quad i > 0 \end{cases}$$

It is indeed a path since, by definition of $E^{(\beta,m)}_{1,k+1,0}$, $\operatorname{supp} P_{k+1}(i_{k+1})$ must be contained in $\operatorname{supp} s$. We then have,

$$\exp t = \frac{\partial R}{\partial P_0[1:]}$$

Moreover, $R \in \mathcal{P}_{\mathbb{L}}\left(\sum_{s\in E^{(\beta,m)}_{1,k,0}} \exp s\right)$. By assumption on x, all the $\{\, P_0[1:] \mid P_0 \in \mathcal{P}_0(x)\}$ is a singleton and so is $\{\, \partial P_0[1:] \mid P_0 \in \mathcal{P}_0(x)\}$. By induction hypothesis and Proposition 3.9.29, $E^{(\beta,m)}_{1,k+1,0}$ has order type less than

$$\omega^{\omega^{\omega(\omega(\mathrm{NR}(x)+\xi+1)a_k+1)}} = a_{k+1}$$

Since the equivalences classes of $\mathcal{P}_{\mathbb{L}}\left(\sum_{s\in E^{(\beta,m)}_{1,k,0}} \exp s\right)/\asymp$ are finite, the ones of $E^{(\beta,m)}_{1,k+1,0}/\asymp$ are also finite. Finally, using Lemmas 3.8.24 and 3.8.19,

$$\begin{aligned}\mathrm{NR}(t) &\leq (\omega \otimes \xi) + \sum_{i=1}^{k_{P_0}-1} \mathrm{NR}(\ln|P_0(i)|) + k_{P_0} + \sum_{j=1}^{k+1}\sum_{i=i_j}^{k_{P_j}-1} \mathrm{NR}\left(\ln|P_j(i)|\right) + \sum_{j=0}^{k+1} \max(0, k_{P_j} - i_j) + 4 \\ &\leq \omega(\mathrm{NR}(x) + \xi + 1)\end{aligned}$$

Then, $$\mathrm{NR}\left(\sum_{t'\in E^{(\beta,m)}_{1,k+1}} \exp t'\right) \leq \omega(\mathrm{NR}(x) + \xi + 1)a_{k+1}$$

We conclude thanks to the induction principle.

(vi) We have $\bigcup_{k\in\mathbb{N}} E^{(\beta,m)}_{1,k,0} \subseteq \left\langle E^{(\beta,m)}_{1,1,0}\right\rangle$. Then, using (v) and applying Proposition 2.4.5, it has order type at most $\omega^{\widehat{a_1}} \leq \omega^{\omega a_1}$.

(vii) We define the following sequence :

- $b_0 = \omega^{\widehat{a_1}}$
- $b_{k'+1} = \omega^{\omega^{\omega(\omega(\mathrm{NR}(x)+\xi+4)b_{k'}+1)}}$

We show that $\bigcup_{k\in\mathbb{N}} E^{(\beta,m)}_{1,k,k'}$ is reverse well-ordered with order type less than $b_{k'}$. We also claim that the equivalence classes of $\bigcup_{k\in\mathbb{N}} E^{(\beta,m)}_{1,k,k'}/\asymp$ are finite and that

$$\mathrm{NR}\left(\sum_{t\in \bigcup_{k\in\mathbb{N}} E^{(\beta,m)}_{1,k,k'}} \exp t\right) \leq \omega(\mathrm{NR}(x) + \xi + 4)b_{k'}$$

We show it by induction on $k' \in \mathbb{N}$.

- For $k' = 0$, we just apply (vi).
- Assume the property for some $k' \in \mathbb{N}$. Let $t \in \bigcup_{k\in\mathbb{N}} E^{(\beta,m)}_{1,k,k'+1}$. Let $(P_0, 0)(P_1, i_1), \dots, (P_{k+k'+1}, i_{k+k'+1})$ minimal for the order $(<_{lex}, <)_{lex}$ such that $t = e^{(\beta,m)}\begin{pmatrix} P_0; P_1, \dots, P_k \\ P_{k+1}, \dots, P_{k+k'+1} \\ i_1, \dots, i_{k+k'+1} \end{pmatrix}$. Then,

$$t = e^{(\beta,m)}\begin{pmatrix} P_0; P_1, \dots, P_k \\ P_{k+1}, \dots, P_{k+k'} \\ i_1, \dots, i_{k+k'} \end{pmatrix} - \sum_{\ell=m+2}^{+\infty} \ln_\ell \kappa_{-\beta} - \sum_{\substack{\gamma > \beta, \ell \in \mathbb{N}^* \\ \gamma \mid \kappa_{-\gamma} \succeq^K P_{k+k'+1}(k_{P_{k+k'+1}})}} \ln_m \kappa_{-\beta} + \sum_{i=i_{k+1}}^{+\infty} \ln|P_{k+1}(i)|$$

Write $s = e^{(\beta,m)}\begin{pmatrix} P_0;P_1,\dots,P_k \\ P_{k+1},\dots,P_{k+k'} \\ i_1,\dots,i_{k+k'} \end{pmatrix}$. We then have,

$$\partial(\ln_{m+1}\kappa_{-\beta})\exp t = \exp(s)\exp\left(-\sum_{\substack{\ell\in\mathbb{N}^* \\ \gamma \mid \kappa_{-\gamma}\succeq^K P_{k+k'+1}(k_{P_{k+k'+1}})}} \ln_m\kappa_{-\beta} + \sum_{i=i_{k+1}}^{+\infty}\ln|P_{k+1}(i)|\right)$$

Consider the following path : $\begin{cases} R(0) = \exp s \\ R(i) = P_{k+k'+1}(i-1+i_{k+1}) \quad i>0 \end{cases}$

It is indeed a path since, by definition of $E^{(\beta,m)}_{1,k,k'+1}$, $\operatorname{supp} P_{k+k'+1}(i_{k+k'+1})$ must be contained in $\operatorname{supp} s$. Then,

$$\partial(\ln_{m+1}\kappa_{-\beta})\exp t = \partial R$$

Moreover, $R\in\mathcal{P}_{\mathbb{L}}\left(\sum_{s\in\bigcup_{k\in\mathbb{N}}E^{(\beta,m)}_{1,k,k'}}\exp s\right)$. By induction hypothesis and Proposition 3.9.29, $\bigcup_{k\in\mathbb{N}}E^{(\beta,m)}_{1,k,k'+1}$ has order type less than

$$\omega^{\omega^{\omega(\omega(\mathrm{NR}(x)+\xi+4)b_{k'}+1)}} = b_{k'+1}$$

Since the equivalences classes of $\mathcal{P}_{\mathbb{L}}\left(\sum_{s\in\bigcup_{k\in\mathbb{N}}E^{(\beta,m)}_{1,k,k'}}\exp s\right)/\asymp$ are finite, the ones of $\bigcup_{k\in\mathbb{N}}E^{(\beta,m)}_{1,k,k'+1}/\asymp$ are also finite. Finally, using Lemmas 3.8.24 and 3.8.19,

$$\begin{aligned}\mathrm{NR}(t) &\le (\omega\oplus\omega\otimes\xi\oplus\omega) + \sum_{j=0}^{k+1}\sum_{i=i_j}^{k_{P_j}-1}\mathrm{NR}\left(\ln|P_j(i)|\right) + \sum_{j=0}^{k+1}\max(0,k_{P_j}-i_j) \\ &\le \omega(\mathrm{NR}(x)+\xi+4)\end{aligned}$$

Then, $$\mathrm{NR}\left(\sum_{t'\in\bigcup_{k\in\mathbb{N}}E^{(\beta,m)}_{1,k,k'+1}}\exp t'\right) \le \omega(\mathrm{NR}(x)+\xi+4)b_{k'+1}$$

We conclude thanks to the induction principle.

(viii) By easy induction, for all $k\in\mathbb{N}$, $b_{k'}<\lambda$.

(ix) Using (iv), we get that for all $N\in\mathbb{N}$, $\bigcup_{k'=0}^{N}\bigcup_{k\in\mathbb{N}}E^{(\beta,m)}_{1,,k,2k'}$ is an initial segment of $\bigcup_{k'\in\mathbb{N}}\bigcup_{k\in\mathbb{N}}E^{(\beta,m)}_{1,k,2k'}$. We also have that $\bigcup_{k'=0}^{N}\bigcup_{k\in\mathbb{N}}E^{(\beta,m)}_{1,k,2k'+1}$ is an initial segment of $\bigcup_{k'\in\mathbb{N}}\bigcup_{k\in\mathbb{N}}E^{(\beta,m)}_{1,k,2k'+1}$. Using (vii), we get that $\bigcup_{k'\in\mathbb{N}}\bigcup_{k\in\mathbb{N}}E^{(\beta,m)}_{1,k,2k'}$ has order type at most

$$\sup\left\{\bigoplus_{k=0}^{N}b_{2k'}\;\middle|\;N\in\mathbb{N}\right\} = \sup\{b_{2N}\mid N\in\mathbb{N}\}\underset{\text{by (viii)}}{\le}\lambda$$

Similarly, $\bigcup_{k'\in\mathbb{N}}\bigcup_{k\in\mathbb{N}}E^{(\beta,m)}_{1,k,2k'+1}$ has order type at most λ. Using Proposition 2.4.2, we conclude that $E^{(\beta,m)}_1$ has order type at most 2λ.

Now we deal with the case $(\beta,m)=(\alpha,n)$. A close looking at point (v) above reveals that the property it shows does not depend on (β,m). Then we have, using a similar argument as in points (viii) and (ix), that $\bigcup_{k\in\mathbb{N}}E^{(\alpha,n)}_{1,k,0}$ has order type at most 2λ. Then, for any $(\beta,m)\le_{lex}(\alpha,n)$, $E^{(\beta,m)}_1$ is reverse well-ordered with order type at most 2λ. Using again Proposition 2.4.2 and the properties of $E^{(\beta,m)}_2$ and $E^{(\beta,m)}_3$ mentioned in the beginning of this proof, we get that $E^{(\beta,m)}$ is reverse well-ordered with order type at most $2\lambda+\omega(\xi+1)$. □

5.2.2 Length of the series of the anti-derivative of an arbitrary surreal number

Proposition 5.2.11. *Let x be a surreal number. Let γ be the smallest ordinal such that $\kappa_{-\gamma}\prec^K P(k_P)$ for all path $P\in\mathcal{P}_{\mathbb{L}}(x)$. Let λ be the least ε-number greater than $\mathrm{NR}(x)$ and γ. Then $\bigcup_{i\in\mathbb{N}}\operatorname{supp}\Phi^i(x)$ is reverse well-ordered with order type less than $\omega^{\omega^{\lambda+2}}$.*

Proof. Let $\alpha < \gamma$ and $n \in \mathbb{N}$. Write $x = \sum_{a \in \operatorname{supp} x} r_a \omega^a$. For any ordinal $\alpha < \gamma$, $n \in \mathbb{N}$, $r \in \mathbb{R} \setminus \{-1\}$ and any term $s\omega^{a_0}$, define

$$S_{\alpha,n,1,s\omega^{a_0}} = \left\{ a \in \operatorname{supp} x \;\middle|\; \exists \varepsilon \in \mathbf{No}_\infty \ \forall (\beta, m) <_{lex} (\alpha, n) \quad \begin{cases} \ln_n \kappa_{-\alpha} \prec \varepsilon \sim s\omega^{a_0} \prec \ln_m \kappa_{-\beta} \\ \wedge \omega^a \asymp \partial(\ln_n \kappa_{-\alpha}) \exp \varepsilon \end{cases} \right\}$$

$$S_{\alpha,n,2,r} = \{ a \in \operatorname{supp} x \mid \exists \varepsilon \in \mathbf{No}_\infty \qquad \varepsilon \sim r \ln_n \kappa_{-\alpha} \wedge \omega^a \asymp \partial(\ln_n \kappa_{-\alpha}) \exp \varepsilon \}$$

$$x_{\alpha,n,1,s\omega^{a_0}} = \sum_{a \in S_{\alpha,n,1,s\omega^{a_0}}} r_a \omega^a \qquad \text{and} \qquad x_{\alpha,n,2,r} = \sum_{a \in S_{\alpha,n,2,r}} r_a \omega^a$$

All theses surreal numbers have disjoint supports and $x = \sum_{s\omega^{a_0} \in \mathbb{R}\omega^{\mathbf{No}}} \sum_{\alpha<\gamma} \sum_{n \in \mathbb{N}} x_{\alpha,n,1,s\omega^{a_0}} + \sum_{r \in \mathbb{R}\setminus\{-1\}} \sum_{\alpha<\gamma} \sum_{n\in\mathbb{N}} x_{\alpha,n,2,r}$. We then study both sums of the above equality.

- The set $\{ r \in \mathbb{R} \setminus \{-1\} \mid S_{\alpha,n,2,r} \neq \varnothing \}$ is reverse well-ordered with order type at most $\nu(x)$. Since Φ is strongly linear,

$$\bigcup_{i\in\mathbb{N}} \operatorname{supp} \Phi^i \left(\sum_{r\in\mathbb{R}\setminus\{-1\}} \sum_{\alpha<\gamma} \sum_{n\in\mathbb{N}} x_{\alpha,n,2,r} \right) \subseteq \bigcup_{r\in\mathbb{R}\setminus\{-1\}} \bigcup_{\alpha<\gamma} \bigcup_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} \operatorname{supp} \Phi^i(x_{\alpha,n,2,r})$$

  Using Corollary 5.2.5, $\bigcup_{i\in\mathbb{N}} \operatorname{supp} \Phi^i(x_{\alpha,n,2,r})$ is reverse well-ordered with order type at most $\omega^{\omega(2\lambda+\omega(\gamma+1)+1)}$. Moreover, Lemma 5.2.1 ensure that if $(\alpha, n, r) >_{lex} (\alpha', n', r')$, then

$$\bigcup_{i\in\mathbb{N}} \operatorname{supp} \Phi^i(x_{\alpha,n,2,r}) < \bigcup_{i\in\mathbb{N}} \operatorname{supp} \Phi^i(x_{\alpha',n',2,r'})$$

  We end up with the fact that $\bigcup_{i\in\mathbb{N}} \operatorname{supp} \Phi^i \left(\sum_{r\in\mathbb{R}\setminus\{-1\}} \sum_{\alpha<\gamma} \sum_{n\in\mathbb{N}} x_{\alpha,n,2,r} \right)$ is reverse well-ordered with order type at most $\omega^{\omega(2\lambda+\omega(\gamma+1)+1)}\nu(x)\gamma$.

- Since Φ is strongly linear,

$$S_1 := \bigcup_{i\in\mathbb{N}} \operatorname{supp} \Phi^i \left(\sum_{s\omega^{a_0} \in \mathbb{R}\omega^{\mathbf{No}}} \sum_{\alpha<\gamma} \sum_{n\in\mathbb{N}} x_{\alpha,n,1,s\omega^{a_0}} \right) \subseteq \bigcup_{s\omega^{a_0} \in \mathbb{R}\omega^{\mathbf{No}}} \bigcup_{\alpha<\gamma} \bigcup_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} \operatorname{supp} \Phi^i(x_{\alpha,n,1,s\omega^{a_0}})$$

  Denote $H^{(\beta,m)}(x_{\alpha,n,1,s\omega^{a_0}})$, $E^{(\beta,m)}(x_{\alpha,n,1,s\omega^{a_0}})$ the sets defined as in Corollary 5.2.9 for $x_{\alpha,n,1,s\omega^{a_0}}$. Then, using this corollary,

$$S_1 \subseteq \bigcup_{\beta<\gamma} \bigcup_{m\in\mathbb{N}} \bigcup_{\substack{\alpha,\, n \mid (\beta,\, m) \leq_{lex} (\alpha,\, n) \\ \alpha < \gamma}} \bigcup_{s\omega^{a_0} \in \mathbb{R}\omega^{\mathbf{No}}} \bigcup_{\substack{\eta \in H^{(\beta,m)}(x_{\alpha,n,1,s\omega^{a_0}}) \\ y \in \langle E^{(\beta,m)}(x_{\alpha,n,1,s\omega^{a_0}}) \rangle}} \operatorname{supp} \left(\frac{\partial \ln_m \kappa_{-\beta}}{\partial \ln_n \kappa_{-\alpha}} \exp(\eta + y) \right)$$

  We also know that $(\alpha, n, s\omega^{a_0}) >_{lex} (\alpha', n', s'\omega^{a_0'})$, then

$$\bigcup_{\substack{\eta \in H^{(\beta,m)}(x_{\alpha,n,1,s\omega^{a_0}}) \\ y \in \langle E^{(\beta,m)}(x_{\alpha,n,1,s\omega^{a_0}}) \rangle}} \operatorname{supp} \left(\frac{\partial \ln_m \kappa_{-\beta}}{\partial \ln_n \kappa_{-\alpha}} \exp(\eta + y) \right) \subsetneq \bigcup_{\substack{\eta \in H^{(\beta,m)}\left(x_{\alpha',n',1,s'\omega^{a_0'}}\right) \\ y \in \left\langle E^{(\beta,m)}\left(x_{\alpha',n',1,s'\omega^{a_0'}}\right) \right\rangle}} \operatorname{supp} \left(\frac{\partial \ln_m \kappa_{-\beta}}{\partial \ln_n \kappa_{-\alpha}} \exp(\eta + y) \right)$$

  Propositions 5.2.10 and 2.4.5 guarantee that all of theses sets are reverse well-ordered with order type less than $\omega^{2\omega\lambda}$. Let

$$S_{\beta,m} = \bigcup_{\substack{\alpha,\, n \mid (\beta,\, m) \leq_{lex} (\alpha,\, n) \\ \alpha < \gamma}} \bigcup_{s\omega^{a_0} \in \mathbb{R}\omega^{\mathbf{No}}} \bigcup_{\substack{\eta \in H^{(\beta,m)}(x_{\alpha,n,1,s\omega^{a_0}}) \\ y \in \langle E^{(\beta,m)}(x_{\alpha,n,1,s\omega^{a_0}}) \rangle}} \operatorname{supp} \left(\frac{\partial \ln_m \kappa_{-\beta}}{\partial \ln_n \kappa_{-\alpha}} \exp(\eta + y) \right)$$

  The set of possible $s\omega^{a_0}$ is reverse well-ordered with order type at most $\nu(x)$. Moreover, α and n are determined from $s\omega^{a_0}$. Then $S_{\beta,m}$ is reverse well-ordered with order type at most $\omega^{2\omega^{\lambda+1}}\nu(x)$. Finally, if $(\beta, m) >_{lex} (\beta', m')$, then $S_{\beta,m} \subsetneq S_{\beta',m'}$ and there are at most $\omega\gamma$ such ordered pairs. Then, S_1 is reverse well-ordered with order type at most $\omega^{2\omega\lambda+1}\nu(x)\gamma$.

Both sets have order type less than $\omega^{\omega^{\lambda+2}}$, which is a multiplicative ordinal. Using Proposition 2.4.3, $\bigcup_{i\in\mathbb{N}} \operatorname{supp} \Phi^i(x)$ has order type less than $\omega^{\omega^{\lambda+2}}$.

□

5.3 Surreal subfields stable under exponential, logarithm, derivative and anti-derivative

We now wonder how to get a surreal field that is stable under $\exp, \ln, \partial$ and anti-derivation and that is not **No** itself. A first try would be to look at admissible sets. Namely, any analog definition of **No** inside an admissible set would be stable by any of these operations. However, such an approach may seem quite disappointing since we do not actively build something new or characterize such a field. In this section, we want to build a subfield of **No** that is stable under all these operations. We even want a lot of such fields.
We can express a sufficient condition to get a field stable under derivation and anti-derivation.

Theorem 5.3.1. *Let α be a limit ordinal and $(\Gamma_\beta)_{\beta<\alpha}$ be a sequence of Abelian subgroups of* **No** *such that*

- $\forall\beta<\alpha \quad \forall\gamma<\beta \qquad \Gamma_\gamma\subseteq\Gamma_\beta$
- $\forall\beta<\alpha \qquad \omega^{(\Gamma_\beta)^*_+}\succ^K \kappa_{-\varepsilon_\beta}$
- $\forall\beta<\alpha \quad \forall\gamma<\varepsilon_\beta \qquad \kappa_{-\gamma}\in\omega^{\Gamma_\beta}$
- $\forall\beta<\alpha \quad \exists\eta_\beta<\varepsilon_\beta \quad \forall x\in\omega^{\Gamma_\beta} \qquad \mathrm{NR}(x)<\eta_\beta$

Then $\bigcup\limits_{\beta<\alpha}\mathbb{R}_{\varepsilon_\beta}^{\Gamma_\beta^{\uparrow\varepsilon_\beta}}$ *is stable under* $\exp, \ln, \partial$ *and anti-derivation.*

Proof. Let $\mathbb{K}=\bigcup\limits_{\beta<\alpha}\mathbb{R}_{\varepsilon_\beta}^{\Gamma_\beta^{\uparrow\varepsilon_\beta}}$ As an increasing union of fields, $\mathbb{K}$ is indeed a field.

(i) Using Theorem 5.1.6, each field $\mathbb{R}_{\varepsilon_\beta}^{\Gamma_\beta^{\uparrow\varepsilon_\beta}}$ is stable under $\exp$ and $\ln$, then so is $\mathbb{K}$.

(ii) Write $\Gamma_\beta^{\uparrow\varepsilon_\beta}=(\Gamma_{i,\beta})_{i<\gamma_{\varepsilon_\beta}}$. We use the notation introduce in the beginning Definition 5.1.5. We prove by induction on $i<\gamma_{\varepsilon_\beta}$ that for $x\in\Gamma_{i,\beta}$, $\mathrm{NR}(\omega^x)<\eta_\beta e_i$.

- For $i=0$ we have $e_0=1$ and $\Gamma_{0,\beta}=\Gamma_\beta$. By assumption on Γ_β, for all $x\in\Gamma_{0,\beta}$, $\mathrm{NR}(\omega^x)<\eta_\beta=\eta_\beta e_0$.
- Assume the property for some ordinal i. Then let $x\in\Gamma_{i+1,\beta}$. Write $x=u+v+\sum\limits_{k=1}^{p}h(w_k)$ with $u\in\Gamma_{i,\beta}$, $v\in\mathbb{R}_{e_i}^{g\left((\Gamma_{i,\beta})^*_+\right)}$ and w_k such $r\omega^{w_k}$ is a term of some element $y_k\in\Gamma_{i,\beta}$, for some $r\in\mathbb{R}$. Using Corollary 3.8.25,
$$\mathrm{NR}(\omega^x)\leq\mathrm{NR}(\omega^u)+\mathrm{NR}(\omega^v)+\sum_{k=1}^{p}\mathrm{NR}(\omega^{h(w_k)})+p+1$$
From induction hypothesis, $\qquad \mathrm{NR}(\omega^u)<\eta_\beta e_i$
Write $v=\sum\limits_{j<\nu}r_j\omega^{g(a_j)}$. Then $\omega^v=\exp\left(\sum\limits_{j<\nu}r_j\omega^{a_j}\right)$. From induction hypothesis, $\mathrm{NR}(\omega^{a_j})<\eta_\beta e_i$. Then $\mathrm{NR}(r_j\omega^{a_j})<\eta_\beta e_i+1$. Then
$$\mathrm{NR}(\omega^v)=\mathrm{NR}\left(\sum_{j<\nu}r_j\omega^{a_j}\right)\leq(\eta_\beta e_i+1)\otimes\nu\leq(\eta_\beta e_i+1)\otimes e_i\leq\eta_\beta e_i^2$$
We also have $\qquad \mathrm{NR}(\omega^{h(w_k)})=\mathrm{NR}(\omega^{\omega^{\omega^{w_k}}})\leq\mathrm{NR}(\omega^{y_k})<\eta_\beta e_i$
Finally, $\qquad \mathrm{NR}(\omega^x)\leq(p+1)(\eta_\beta e_i+1)+\eta_\beta e_i^2<\eta_\beta e_{i+1}$
- Assume i is a limit ordinal. Then by definition of $\Gamma_{i,\beta}$ for any $x\in\Gamma_{i,\beta}$ there is some $j<i$ such that $x\in\Gamma_{j,\beta}$. Then induction hypothesis concludes.

(iii) Let $x\in\mathbb{K}$ and $\beta<\alpha$ such that $x\in\mathbb{R}_{\varepsilon_\beta}^{\Gamma_\beta^{\uparrow\varepsilon_\beta}}$. Using (ii), there is $i<\gamma_{\varepsilon_\beta}$ such that
$$\mathrm{NR}(x)\leq(\eta_\beta e_i+1)\otimes\nu(x)<\eta_\beta\otimes\varepsilon_\beta=\varepsilon_\beta$$
Since $\eta_\beta\otimes\varepsilon_\beta$ is a limit ordinal, then we also have $\mathrm{NR}(x)+1<\eta_\beta\otimes\varepsilon_\beta=\varepsilon_\beta$.

(iv) Let $x\in\mathbb{K}$ and $\beta<\alpha$ such that $x\in\mathbb{R}_{\varepsilon_\beta}^{\Gamma_\beta^{\uparrow\varepsilon_\beta}}$. Using (iii) and Proposition 3.9.29, $\nu(\partial x)<\omega^{\omega^{\omega(\mathrm{NR}(x)+1)}}<\varepsilon_\beta$. Using Corollary 5.1.13 and (i), we also have for all $P\in\mathcal{P}(x)$, $\partial P\in\mathbb{R}\omega^{\Gamma_\beta^{\uparrow\varepsilon_\beta}}$. Then,
$$\partial x\in\mathbb{R}_{\varepsilon_\beta}^{\Gamma_\beta^{\uparrow\varepsilon_\beta}}\subseteq\mathbb{K}$$
Then $\mathbb{K}$ is stable under ∂.

(v) Let $x \in \mathbb{K}$ and $\beta < \alpha$ such that $x \in \mathbb{R}_{\varepsilon_\beta}^{\Gamma_\beta^{\uparrow\varepsilon_\beta}}$. Using Proposition 5.2.11 and the definition of $\mathcal{A}$,

$$\nu\left(\mathcal{A} \circ \left(\sum_{i\in\mathbb{N}} \Phi^i\right)(x)\right) < \omega^{\omega^{\lambda+2}}$$

where λ is least ε-number greater $\mathrm{NR}(x)$ and such that

$$\forall P \in \mathcal{P}_{\mathbb{L}}(x) \qquad \kappa_{-\lambda} \prec^K P(k_P)$$

Using (iii), $\mathrm{NR}(x) < \varepsilon_\beta$. Let $P \in \mathcal{P}_{\mathbb{L}}(x)$. Using (i), $\mathbb{R}_{\varepsilon_\beta}^{\Gamma_\beta^{\uparrow\varepsilon_\beta}}$ is stable under $\exp$ and ln. Since $P(i+1)$ is a term of $\ln|P(i)|$, if $P(i) \in \omega^{\Gamma_\beta}$, then $P(i+1) \in \omega^{\Gamma_\beta}$. By induction, $P(k_P) \in \omega^{\Gamma_\beta}$. Since $P(k_P)$ is infinitely large, $P(k_P) \in \omega^{(\Gamma_\beta)_+^*}$. By assumption on Γ_β, $P(k_P) \succ^K \kappa_{-\varepsilon_\beta}$. Finally, $\lambda \leq \varepsilon_\beta$ and

$$\nu\left(\mathcal{A} \circ \left(\sum_{i\in\mathbb{N}} \Phi^i\right)(x)\right) < \omega^{\omega^{\varepsilon_\beta+2}} < \varepsilon_{\beta+1}$$

Propositions 5.2.2 and 5.2.8 and the third assumption about Γ_β ensure that each term of $\mathcal{A} \circ \left(\sum_{i\in\mathbb{N}} \Phi^i\right)(x)$ is in $\omega^{\Gamma_\beta} \subseteq \omega^{\Gamma_{\beta+1}}$. Then

$$\mathcal{A} \circ \left(\sum_{i\in\mathbb{N}} \Phi^i\right)(x) \in \mathbb{R}_{\varepsilon_{\beta+1}}^{\Gamma_{\beta+1}^{\uparrow\varepsilon_{\beta+1}}}$$

Application of Corollary 3.10.14 gives that $\mathbb{K}$ is stable under anti-derivation.

□

In a previous version of this thesis, there was an example of application of this theorem. When reviewing this thesis, Mickaël Matusinski pointed out we can make use of the idea presented in this example make Theorem 5.3.1 even stronger, removing the last condition. This is a noticeable improvement since this last condition looks quite *ad hoc*.

Corollary 5.3.2 (Guilmant-Matusinski)**.** *Let α be a limit ordinal and $(\Gamma_\beta)_{\beta<\alpha}$ be a sequence of Abelian subgroups of* **No** *such that*

- $\forall\beta < \alpha \quad \forall\gamma < \beta \qquad \Gamma_\gamma \subseteq \Gamma_\beta$
- $\forall\beta < \alpha \qquad \omega^{(\Gamma_\beta)_+^*} \succ^K \kappa_{-\varepsilon_\beta}$
- $\forall\beta < \alpha \quad \forall\gamma < \varepsilon_\beta \qquad \kappa_{-\gamma} \in \omega^{\Gamma_\beta}$

Then $\bigcup_{\beta<\alpha} \mathbb{R}_{\varepsilon_\beta}^{\Gamma_\beta^{\uparrow\varepsilon_\beta}}$ *is stable under* $\exp$, $\ln$, ∂ *and anti-derivation.*

This corollary is just Theorem 5.3.1 with the last hypothesis dropped.

Proof. Let $(\Gamma_\beta)_{\beta<\alpha}$ be a sequence of Abelian subgroups of **No** as above. Let $\left(\eta'_\beta\right)_{\beta<\alpha}$ be any increasing sequence of ordinals such that $\sup\{\eta'_\beta\}_{\beta<\alpha} = \varepsilon_\alpha$ and $\eta'_\beta < \varepsilon_\beta$ for all $\beta < \alpha$. We define for $\beta < \alpha$

$$\Gamma'_\beta = \left\{x \in \Gamma_\beta \;\middle|\; \mathrm{NR}(\omega^x) < \omega^{\eta'_\beta}\right\}$$

Note that since $\omega^{\eta'_\beta}$ is an additive ordinal, and using Corollary 3.8.25, Γ'_β is indeed an Abelian group. Let $\gamma < \beta < \alpha$. Then $\Gamma'_\gamma \subseteq \left\{x \in \Gamma_\beta \;\middle|\; \mathrm{NR}(x) < \omega^{\eta'_\gamma}\right\} \subseteq \Gamma'_\beta$ and

$$\omega^{\left(\Gamma'_\beta\right)_+^*} \subseteq \omega^{(\Gamma_\beta)_+^*} \succ^K \kappa_{\varepsilon_\beta}$$

Therefore,

$$\omega^{\left(\Gamma'_\beta\right)_+^*} \succ^K \kappa_{\varepsilon_\beta}$$

Note also that for $\beta < \alpha$ and $\gamma < \varepsilon_\beta$, $\mathrm{NR}(\kappa_{-\gamma}) = 0 < \omega^{\eta'_\beta}$ and that $\kappa_{-\gamma} \in \Gamma_\beta$. Hence, $\kappa_{-\gamma} \in \Gamma'_\beta$. Finally, taking $\eta_\beta = \omega^{\eta'_\beta}$ we get that we can apply Theorem 5.3.1 and get that $\bigcup_{\beta<\alpha} \mathbb{R}_{\varepsilon_\beta}^{\left(\Gamma'_\beta\right)^{\uparrow\varepsilon_\beta}}$ is stable under $\exp$, $\ln$, ∂ and anti-derivation. Noting that we have

$$\bigcup_{\beta<\alpha} \mathbb{R}_{\varepsilon_\beta}^{\left(\Gamma'_\beta\right)^{\uparrow\varepsilon_\beta}} = \bigcup_{\beta<\alpha} \mathbb{R}_{\varepsilon_\beta}^{\Gamma_\beta^{\uparrow\varepsilon_\beta}}$$

we get the desired result. □

I finish this section with an example[1] of application.

Example 5.3.3. Take $\alpha = \omega$ and for $n < \omega$, $\Gamma_n = \mathbf{No}_{\varepsilon_n}$. We first recall that from Lemma 3.7.21, for any ordinal α,

$$\kappa_{-\alpha} = \omega^{\omega^{-\omega\otimes\alpha}}$$

in particular

$$\kappa_{-\varepsilon_n} = \omega^{\omega^{-\omega\otimes\varepsilon_n}} = \omega^{\omega^{-\varepsilon_n}} = \omega^{\frac{1}{\varepsilon_n}}$$

From Theorem 3.3.28, we know that the signs sequence of $\omega^{-\omega\otimes\alpha}$ is $(+)(-)^{\omega\otimes\alpha}$, which has length $1 \oplus \omega \otimes \alpha$.

- Since ε_n is an ε-number, hence an additive ordinal, using Theorem 3.4.1, for any $n \in \mathbb{N}$, Γ_n is an Abelian group.
- Of course for any $n \leq m$, $\Gamma_n \subseteq \Gamma_m$.
- Since $|\omega^{-\varepsilon_n}|_{+-} = 1 \oplus \omega \otimes \varepsilon_n = \varepsilon_n$, we have $\omega^{-\varepsilon_n} < (\Gamma_n)^*_+$ and thus $\kappa_{-\varepsilon_n} \prec \omega^{\Gamma_n}$.
- Also, for all $\gamma < \varepsilon_n$ we have $1 \oplus \omega \otimes \gamma < \varepsilon_n$ hence $\kappa_{-\gamma} = \omega^{\omega^{-\omega\otimes\alpha}} \in \omega^{\mathbf{No}_{\varepsilon_n}}$.

Theorem 5.3.1 applies and $\bigcup_{n\in\mathbb{N}} \mathbb{R}_{\varepsilon_n}^{\Gamma_n^{\uparrow\varepsilon_n}}$ is stable under $\exp$, $\ln$, ∂ and anti-derivation. As a final note, we can notice that

$$\bigcup_{n\in\mathbb{N}} \mathbb{R}_{\varepsilon_n}^{\Gamma_n^{\uparrow\varepsilon_n}} = \bigcup_{n\in\mathbb{N}} \mathbb{R}_{\varepsilon_n}^{\mathbf{No}_{\varepsilon_n}^{\uparrow\varepsilon_n}}$$

[1] This is the example mentioned before the statement of the corollary but updated.

Chapter 6

Topological aspects

Surreal numbers can be seen as "all the numbers" according to Conway's idea. However, it does not mean that this is a continuum. In fact, even if we build more and more numbers, the way we proceed naturally builds many holes or gaps in the surreal line. This leads to some problem regarding the topology and the continuity of the functions. For instance, do we want the following function to be continuous ?

$$f : \begin{cases} \mathbb{R}_\lambda^\Gamma & \to & \mathbb{R}_\lambda^\Gamma \\ x & \mapsto & \begin{cases} 0 & \text{if } \exists n \in \mathbb{N} \qquad x \leq n \\ 1 & \text{otherwise} \end{cases} \end{cases}$$

In fact this function is indeed continuous considering the usual definition of the continuity:

Definition 6.0.1. $f : \mathbb{R}_\lambda^\Gamma \to \mathbb{R}_\lambda^\Gamma$ is continuous if the following holds:

$$\forall x \in \mathbb{R}_\lambda^\Gamma \quad \forall \varepsilon \in \left(\mathbb{R}_\lambda^\Gamma\right)_+^* \quad \exists \eta \in \left(\mathbb{R}_\lambda^\Gamma\right)_+^* \quad \forall y \in \mathbb{R}_\lambda^\Gamma \qquad |x-y| < \eta \implies |f(x) - f(y)| < \varepsilon$$

However, this function does not satisfy the Intermediate Value Theorem: there is no x such that $f(x) = \dfrac{1}{2}$ whereas $f(0) = 0$ and $f(\omega) = 1$. Therefore, the expected way to define continuity is not satisfying in the context of surreal numbers.

This chapter introduces the concept of gap-continuity, a special kind of continuity that ensures to get both the Intermediate Value Theorem and the Extreme Value Theorem. To do so, we must at least work on fields where a bounded sequence is guaranteed to converge.

- Section 6.1 studies the Cauchy-completions of surreal fields. Namely, it takes a close look at the condition needed on the surreal fields to guarantee their Cauchy sequences to converge.
- Section 6.2 introduces the gap-compactness, a reinforcement of the notion of compactness for real numbers. This is a notion that requires sets to behave nicely with non-Cauchy gaps.
- Finally, Section 6.3 introduces the notion of gap continuity and uses gap compactness properties to show when we can apply the Intermediate Value Theorem and the Extreme Value Theorem.

The main results of this chapter are two propositions and one theorem.

- Proposition 6.2.10 characterizes gap-compact sets as closed, bounded and gap-connected sets.
- Proposition 6.2.16 give an analogous definition of gap-compactness as Borel's definition of compact sets.
- Theorem 6.3.8 is the Intermediate Value Theorem in the context of surreal numbers. It requires a stronger notion of continuity than the expected one.

6.1 Cauchy-completions

In this section we investigate some topological properties of surreal numbers an look at the completions of surreal fields. Surreal numbers have a lot of gaps that make topological studies harder. We provide some points to handle this issues.

6.1.1 Gaps

Definition and examples

Informally, a gap in a field $\mathbb{K}$ is an "empty region" of the field, were some element is lacking. For instance, there is a gap between $\{ q \in \mathbb{Q} \mid q < \sqrt{2} \}$ and $\{ q \in \mathbb{Q} \mid q > \sqrt{2} \}$ in $\mathbb{Q}$: $\sqrt{2}$ is not rational. In **No** there is a gap between all appreciable number an positive infinite surreal numbers : no number is not infinite and infinite at the same time. However, this example is quite different from the previous one. Indeed, an infinite number is infinitely far from any appreciable number whereas we can find $q, q' \in \mathbb{Q}$ arbitrarily close (in $\mathbb{Q}$) such that $q < \sqrt{2} < q'$. Therefore we can identify several type of gaps.

Definition 6.1.1 (Gap)**.** Let $\mathbb{K}$ be an ordered field. Consider $\mathcal{K}$ the set of ordered pairs (L, R) of subsets $L, R \subseteq \mathbb{K}$ such that

(i) $L < R$

(ii) $\forall x \in \mathbb{K} \qquad (\exists l \in L \quad x < l) \vee (\exists r \in R \quad r < x)$

Consider the equivalence relation $\equiv$ on $\mathcal{K}$ by $(L, R) \equiv (L', R')$ *iff*

(i) $\forall x \in L \quad \exists x' \in L' \quad \exists x'' \in L \qquad x \leq x' \leq x''$

(ii) $\forall x \in R \quad \exists x' \in R' \quad \exists x'' \in R \qquad x'' \leq x' \leq x$

A **gap** is an element of $\mathcal{K}/\equiv$. We denote by $\langle L \dashv\vdash R \rangle$ the equivalence class of (L, R) in $\mathcal{K}$ an call it the **gap formed** by L and R. We may easily extend the order of $\mathbb{K}$ on $\mathbb{K} \cup \mathcal{K}/\equiv$. We denote $\mathcal{G}\mathbb{K}$ the set of gaps of $\mathbb{K}$.

Remark 6.1.2. Notice that if $(L, R), (L', R') \in \mathcal{K}$ are such that $\langle L \dashv\vdash R \rangle = \langle L' \dashv\vdash R' \rangle$, then for all $x \in \mathbf{No}$, we have

$$x = \max L \Leftrightarrow x = \max L' \qquad \text{and} \qquad x = \min R \Leftrightarrow x = \min R'$$

where the equality does not holds if a term is not defined. In particular, if $\max L$ exists then $\max L'$ also exists and is equal to $\max L$. Similarly, if $\min R$ exists then $\min R'$ also exists and is equal to $\min R$.

Example 6.1.3. Let $\mathbb{R}^{\Gamma}_{\lambda}$ be a surreal field.

- $\langle \mathbb{R} \dashv\vdash \{ x \in \mathbb{R}^{\Gamma}_{\lambda} \mid x > \mathbb{R} \} \rangle$ is a gap that we may denote $+\infty$; it is the gap of positive infinity.
- Similarly, $-\infty = \langle \{ x \in \mathbb{R}^{\Gamma}_{\lambda} \mid x < \mathbb{R} \} \dashv\vdash \mathbb{R} \rangle$ is the gap of negative infinity.
- $\dfrac{1}{+\infty} = \langle \{ x \in \mathbb{R}^{\Gamma}_{\lambda} \mid x \prec 1 \} \dashv\vdash \mathbb{R}^*_+ \rangle$ is the gap of positive infinitesimals.
- $\dfrac{1}{-\infty} = \langle \mathbb{R}^*_- \dashv\vdash \{ x \in \mathbb{R}^{\Gamma}_{\lambda} \mid x \prec 1 \} \rangle$ is the gap of negative infinitesimals.
- **On** $= \left\langle \widetilde{\mathbb{R}^{\Gamma}_{\lambda}} \dashv\vdash \varnothing \right\rangle$ and **Off** $= \left\langle \varnothing \dashv\vdash \widetilde{\mathbb{R}^{\Gamma}_{\lambda}} \right\rangle$ are with respect to $\widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$ what are the gaps $+\infty$ and $-\infty$ with respect to $\mathbb{R}$.

Special kinds of gaps: trivial gaps and Cauchy-gaps

Among gaps there are two special kinds: trivial gaps and Cauchy-gaps. The first one are unavoidable and the second one are easy to handle.

Definition 6.1.4 (Trivial gap)**.** Let $\mathbb{K}$ be an ordered field. A **trivial gap** of $\mathbb{K}$ is a gap $\langle L \dashv\vdash R \rangle$ such that either L has a maximal element or R has a minimal element. We denote by $\mathcal{G}_{\top}\mathbb{K}$ the set of trivial gaps of $\mathbb{K}$ and by $\mathcal{G}_{\perp}\mathbb{K}$ the set of non-trivial gaps of $\mathbb{K}$.

Trivial gaps are always here. They just say that there are missing elements between some element $a \in \mathbb{K}$ and everything above it or between a and everything below it. It is an unavoidable *ad hoc* concept: if we add an element to in between and take the completion under the field operations, there will be a new trivial gap that satisfies the same definition in this new context. For instance $\langle 0 \dashv\vdash \mathbb{R}^*_+ \rangle$ is a gap in $\mathbb{R}$. We can add $\dfrac{1}{\omega} = [0 \mid \mathbb{R}^*_+]$. However there is a new gap, which is very similar: $\left\langle 0 \dashv\vdash \mathbb{R} \left(\dfrac{1}{\omega} \right)^*_+ \right\rangle$. There is no hope to fill this type of gap. But it is not really a problem since we make this kind of gap concrete only when we define them. Typically, there is no way to make a sequence of the considered field that converge to this gap. Everything works as if the field was blind to this kind of gap.
Non-trivial gaps are much more interesting: a sequence can converge to them. For instance, it is perfectly possible to define a sequence in $\mathbb{Q}(\omega)$ which converges to $\sqrt{2}$ and another one that converges to $+\infty < \omega$. However, it may seems intuitive that the gap to $\sqrt{2}$ and the gap $+\infty$ behave much differently. More precisely, among non-trivial gaps, we can distinguish to subcategories: Cauchy-gaps and non-Cauchy-gaps.

Definition 6.1.5 (Cauchy gap)**.** Let $\mathbb{K}$ be an ordered field. A Cauchy-gap of $\mathbb{K}$, is formed by to subsets $L, R \subseteq \mathbb{K}$ such that

(i) We have a gap $\langle L \dashv\vdash R\rangle$

(ii) L has no maximum and R has no minimum (*i.e* $\langle L \dashv\vdash R\rangle \in \mathcal{G}_\perp\mathbb{K}$)

(iii) $\forall \varepsilon \in \mathbb{K}^*_+ \quad \exists l \in L \quad \exists r \in R \qquad r - l < \varepsilon$

Remark 6.1.6.

- A Cauchy-gap is in particular non-trivial.
- Cauchy gaps are gaps that can be filled by completion: Roughly speaking, it lacks just one element to make the field complete.
- On the contrary and similarly to trivial gaps, non-Cauchy-gaps are impossible to fill. For instance, considering the gap $+\infty$ of $\mathbb{R}^\Gamma_\lambda$, we can add a surreal number that is in between $\mathbb{R}$ and $\left\{ x \in \left(\mathbb{R}^\Gamma_\lambda\right)^*_+ \;\middle|\; x \succ 1\right\}$ and take the closure to get a field $\mathbb{S}$. However there is still a new gap $+\infty$ in $\mathbb{S}$ and we did not really solve the problem.

6.1.2 Normal form for gaps

The simplest element that may contribute to fill an arbitrary gap $\langle L \dashv\vdash R\rangle$ of a field of surreal numbers $\mathbb{K}$, if we look at it as subfield of **No**, is just the surreal number $[\,L \mid R]$. As well as this surreal number has a normal form, we can give normal form to the gap $\langle L \dashv\vdash R\rangle$ in $\mathbb{K}$ no matter if it is trivial or not, Cauchy or not. This fact comes from the following observation by Conway on the whole class **No**:

Proposition 6.1.7 ([18, Conway, Chapter 3])**.** *Let $\langle L \dashv\vdash R\rangle$ a gap in* **No** *(L or R must be a proper classes). There are two kind of gaps:*

(Type1) There is a decreasing sequence of surreal number $(a_i)_{i\in\mathbf{Ord}}$ and a sequence of non-zero real numbers $(r_i)_{i\in\mathbf{Ord}}$ such that for any $\nu \in$ **Ord***, there are $x \in L$ and $y \in R$ such that*

$$\sum_{i<\nu} r_i\omega^{a_i} \sqsubset x \qquad \text{and} \qquad \sum_{i<\nu} r_i\omega^{a_i} \sqsubset y$$

In that case we denote the gap $\langle L \dashv\vdash R\rangle$ as $\sum\limits_{i\in\mathbf{Ord}} r_i\omega^{a_i}$.

(Type2) There is some $x = \sum\limits_{i<\nu} r_i\omega^{a_i}$ and a gap $\langle L' \dashv\vdash R'\rangle$ such that

- *For all $a \in \operatorname{supp} x$ there is some $r' \in R'$ such that $r' \leq a$*
- *One of following occurs:*
  - ➢ *L' has no maximum, R' has no minimum and for all $r' \in R'$ there is $r \in R$ such that $r < x + \omega^{r'}$ and for all $l' \in L$ there is some $l \in L$ such that $x + \omega^{l'} < l$. In that case we write*
  $$\langle L \dashv\vdash R\rangle = x + \omega^{\langle L' \dashv\vdash R'\rangle}$$
  - ➢ *L' has no maximum, R' has no minimum and for all $l' \in L'$ there is $r \in R$ such that $r < x - \omega^{l'}$ and for all $r' \in R$ there is some $l \in L$ such that $x - \omega^{r'} < l$. In that case we write*
  $$\langle L \dashv\vdash R\rangle = x - \omega^{\langle L' \dashv\vdash R'\rangle}$$
  - ➢ *R' has a minimum r_0 and for any $r < r_0$ there is some positive $l \in L'$ such that $l \asymp \omega^r$. In that case we write*
  $$\langle L \dashv\vdash R\rangle = x + \omega^{\langle L' \dashv\vdash R'\rangle}$$
  - ➢ *L' has a maximum l_0 and for any $l > l_0$ there is some negative $r \in R'$ such that $r \asymp \omega^l$. In that case we write*
  $$\langle L \dashv\vdash R\rangle = x - \omega^{\langle L' \dashv\vdash R'\rangle}$$

Proof. Let $\langle L \dashv\vdash R\rangle$ be a gap in **No**. If $R = \varnothing$ then $\langle L \dashv\vdash R\rangle = \mathbf{On}$. It is quite easy to see that $\mathbf{On} = \omega^{\mathbf{On}}$ hence is already in normal form. If $L = \varnothing$, then $\langle L \dashv\vdash R\rangle = \mathbf{Off} = -\mathbf{On} = -\omega^{\mathbf{On}}$. In these two cases, we have a normal form. Now assume that $L, R \neq \varnothing$.

- If there is a longest surreal number x such that there are $l \in L$ and $r \in R$ such that $x \trianglelefteq_0 l$ and $x \trianglelefteq_0 r$, consider $L' = L - x$ and $R' = R - x$.

➢ Assume there is some a such that there are $l \in L'$ and $r \in R'$ such that $l \asymp r \asymp \omega^a$. Let $A = \{ u \in \mathbb{R} \mid \exists l \in L \quad u\omega^a \sim l \}$ and $B = \{ u \in \mathbb{R} \mid \exists r \in R \quad u\omega^a \sim r \}$. We clearly have $A < B$. If $[A \mid B] \in \mathbb{R}$ then $[L \mid R] = x + [A \mid B]\omega^a$ which contradicts the fact that $\langle L \dashv\vdash R \rangle$ is a gap. Therefore $[A \mid B] \notin \mathbb{R}$. By assumption on x, neither A nor B is empty. That means that either A has a maximal element, either B has a minimal element.

∵ If A has a maximal element, denoted by u, then $\langle L \dashv\vdash R \rangle = x + u\omega^a + \omega^{\langle \{ b \in \mathbf{No} \mid b < a \} \dashv\vdash a \rangle}$.

∵ If B has a minimal element, denoted by u, then $\langle L \dashv\vdash R \rangle = x + u\omega^a - \omega^{\langle a \dashv\vdash \{ b \in \mathbf{No} \mid b > a \} \rangle}$.

➢ Assume now that there is no a such that there are $l \in L'$ and $r \in R'$ such that $l \asymp r \asymp \omega^a$. Let

$$L'' = \{ a \in \mathbf{No} \mid \exists l \in L' \quad \omega^a \asymp l \} \qquad \text{and} \qquad R'' = \{ a \in \mathbf{No} \mid \exists r \in R' \quad \omega^a \asymp r \}$$

We have either $L'' < R''$ or $R'' < L''$.

∵ If $L'' < R''$, then $\langle L'' \dashv\vdash R'' \rangle$ is gap otherwise $[L \mid R] = x + \omega^{[L'' \mid R'']}$ which contradicts the fact that $\langle L \dashv\vdash R \rangle$ is a gap. We have $\langle L \dashv\vdash R \rangle = x + \omega^{\langle L'' \dashv\vdash R'' \rangle}$.

∵ If $R'' < L''$, then, similarly $\langle R'' \dashv\vdash L'' \rangle$ is gap and we have $\langle L \dashv\vdash R \rangle = x - \omega^{\langle R'' \dashv\vdash L'' \rangle}$.

- If not there no such a longest x. Then, by transfinite induction we build $(x_i)_{i \in \mathbf{Ord}}$ such that for any ordinal numbers $i < j$, $x_i \lhd_0 x_j$. Without loss of generality we can assume $\nu(x_i) = i$. Denoting

$$x_i = \sum_{j<i} r_j \omega^{a_j}$$

we then have $\langle L \dashv\vdash R \rangle = \sum_{i \in \mathbf{Ord}} r_i \omega^{a_i}$.

□

The above proposition extends to the surreal field $\mathbb{R}^{\Gamma}_{\lambda}$ as follows:

Proposition 6.1.8. *Let Γ be an Abelian additive subgroup of* **No**. *Let λ be an ε-number. The gaps in $\mathbb{R}^{\Gamma}_{\lambda}$ are of the following form:*

(Type1) There is a decreasing sequence of surreal number $(a_i)_{i<\lambda}$ and a sequence of non-zero real numbers $(r_i)_{i<\lambda}$ such that for any $\nu < \lambda$, there are $x \in L$ and $y \in R$ such that

$$\sum_{i<\nu} r_i \omega^{a_i} \sqsubset x \qquad \textit{and} \qquad \sum_{i<\nu} r_i \omega^{a_i} \sqsubset y$$

In that case we denote $\langle L \dashv\vdash R \rangle = \sum_{i<\lambda} r_i \omega^{a_i}$. Note that it is surreal number (i.e an element of **No***) but it is not in $\mathbb{R}^{\Gamma}_{\lambda}$. Indeed, the length of the series is λ, which is forbidden in $\mathbb{R}^{\Gamma}_{\lambda}$.*

(Type2) There is some $x = \sum_{i<\nu} r_i \omega^{a_i}$ and a gap $\langle L' \dashv\vdash R' \rangle$ (of Γ) such that

- *For all $a \in \operatorname{supp} x$ there is some $r' \in R'$ such that $r' \leq a$*
- *One of following occurs:*

➢ *L' has no maximum, R' has no minimum and for all $r' \in R'$ there is $r \in R$ such that $r < x + \omega^{r'}$ and for all $l' \in L$ there is some $l \in L$ such that $x + \omega^{l'} < l$. In that case we write*

$$\langle L \dashv\vdash R \rangle = x + \omega^{\langle L' \dashv\vdash R' \rangle}$$

➢ *L' has no maximum, R' has no minimum and for all $l' \in L'$ there is $r \in R$ such that $r < x - \omega^{l'}$ and for all $r' \in R$ there is some $l \in L$ such that $x - \omega^{r'} < l$. In that case we write*

$$\langle L \dashv\vdash R \rangle = x - \omega^{\langle L' \dashv\vdash R' \rangle}$$

➢ *R' has a minimum r_0 and for any $r < r_0$ there is some positive $l \in L'$ such that $l \asymp \omega^r$. In that case we write*

$$\langle L \dashv\vdash R \rangle = x + \omega^{\langle L' \dashv\vdash R' \rangle}$$

➢ *L' has a maximum l_0 and for any $l > l_0$ there is some negative $r \in R'$ such that $r \asymp \omega^l$. In that case we write*

$$\langle L \dashv\vdash R \rangle = x - \omega^{\langle L' \dashv\vdash R' \rangle}$$

The proof is identical to the one of Proposition 6.1.7.

6.1.3 Cauchy completion

As we saw in Remark 6.1.6, it is quite hopeless get rid of all the gaps. Namely if $\langle L \dashv\vdash R\rangle$ is a gap such that for any $l \in L$ and $r \in R$, $r - l$ is infinite, then is impossible do add only one element in between that will not be infinitely far from either L or R. It is also impossible to fill a trivial gap $\langle L \dashv\vdash R\rangle$, for which either L has a maximal element or R has a minimal element. However we can fill the Cauchy gaps and get a Cauchy-completion. This concept is studied in details in Dales and Woodin's book [19].

Definition 6.1.9 (Cauchy complete field)**.** An ordered field $\mathbb{K}$ is **Cauchy complete** is it has no Cauchy gap.

This notion of complete field has to be distinguished from Dedekind's definition of a complete field.

Definition 6.1.10 (Dedekind complete field)**.** An ordered field $\mathbb{K}$ is **Dedekind complete** if any gap of $\mathbb{K}$ is a trivial gap.

In a Dedekind complete field, only the gaps that we cannot avoid are allowed. In particular

Lemma 6.1.11. *Every Dedekind complete field is Cauchy complete.*

Proof. Assume that this does not hold. Then there is trivial gap $\langle L \dashv\vdash R\rangle$ which is a Cauchy gap. Since $\langle L \dashv\vdash R\rangle$ is a trivial gap, either L has a maximal element, either R has a minimal element. This is a contradiction with the definition of a Cauchy gap. □

Cauchy complete field have, as expected, a characterization in terms of Cauchy sequences.

Definition 6.1.12 (Cauchy sequence)**.** Let $\mathbb{K}$ be an ordered field. Let α be its cofinality (*i.e* the smallest ordinal such that there is an increasing function $\varphi : \alpha \to \mathbb{K}$ whose image is cofinal with $\mathbb{K}$). Note that since the inverse function is bijective decreasing from $\mathbb{K}_+^*$ to $\mathbb{K}_+^*$, the ordinal α is also the coinitiality of $\mathbb{K}_+^*$ (*i.e* the cofinality of $-\mathbb{K}_+^* = \mathbb{K}_-^*$). A **Cauchy sequence** of a $\mathbb{K}$ is a sequence $(x_i)_{i<\alpha}$ such that

$$\forall \varepsilon \in \mathbb{K}_+^* \quad \exists i_0 < \alpha \quad \forall i, j > i_0 \qquad |x_i - x_j| < \varepsilon$$

Lemma 6.1.13. *Let $\mathbb{K}$ be an ordered field. The Cauchy sequences of $\mathbb{K}$ converge iff $\mathbb{K}$ is Cauchy-complete.*

Proof. $\overset{\text{NC}}{\Rightarrow}$ Assume there is a Cauchy-gap $\langle L \dashv\vdash R\rangle$. Let α be the coinitiality of $\mathbb{K}_+^*$. Either the cofinality L is α, or R has coinitiality α. For instance, assume that L has cofinality α. Let $(x_i)_{i<\alpha}$ be an increasing cofinal sequence of L. Let $\varepsilon \in \mathbb{K}_+^*$. By definition of a Cauchy gap, there is $l \in L$ and $r \in R$ such that $r - l < \varepsilon$. Since $(x_i)_{i<\alpha}$ is a cofinal increasing sequence, there is a rank $i_0 < \alpha$ such that for all ordinal i such that $i_0 \le i < \alpha$, $l < x_i < r$. In particular, for all $i, j \ge i_0$ $|x_i - x_j| < \varepsilon$. Therefore, $(x_i)_{i<\alpha}$ is a Cauchy sequence. Assume it converges to x. Then by cofinality of the sequence, we must have $x > L$. Since it we have a Cauchy gap $\langle L \dashv\vdash R\rangle$, there must be some $r \in R$ such that $L < r < x$ which contradicts the convergence. Hence, $(x_i)_{i<\alpha}$ is a non-convergent Cauchy sequence.

$\overset{\text{SC}}{\Leftarrow}$ If the have a Cauchy sequence $(x_\alpha)_{\alpha<\beta}$ that does not converge, we consider

$$\begin{aligned} L_\varepsilon &= \{\, x_\alpha - 2\varepsilon \mid \forall \gamma, \delta \quad (\alpha \le \gamma, \delta < \beta) \implies |x_\gamma - x_\delta| < \varepsilon\} \\ R_\varepsilon &= \{\, x_\alpha + 2\varepsilon \mid \forall \gamma, \delta \quad (\alpha \le \gamma, \delta < \beta) \implies |x_\gamma - x_\delta| < \varepsilon\} \end{aligned}$$

and

$$L = \bigcup_{0<\varepsilon<1} L_\varepsilon \qquad \text{and} \qquad R = \bigcup_{0<\varepsilon<1} R_\varepsilon$$

We indeed have $L < R$. If $[\,L \mid R] \in \mathbb{K}$ then it must be arbitrarily close to x_α for α sufficiently large. That means that it must be the limit of the Cauchy sequence. In other words, $[\,L \mid R]$ is a Cauchy Gap.

□

Definition 6.1.14 (Cauchy completion)**.** Let $\mathbb{K}$ an ordered field. The **Cauchy completion** of $\mathbb{K}$ is the smallest subfield $\widetilde{\mathbb{K}} \supseteq \mathbb{K}$ that is Cauchy-complete.

Theorem 6.1.15 ([19, Theorem 3.11])**.** *Let S be a totally ordered set and $\mathcal{F}(\mathbb{R}, S)$ be the set of functions from S to $\mathbb{R}$ with well-ordered support*[1]*. Let G be a sub-group of $\mathcal{F}(\mathbb{R}, S)$ containing $\mathbb{R}\mathbb{1}_{s_0}$ if $s_0 = \min S$. The Cauchy-completion $\widetilde{G}$ of G is the set of functions of $\mathcal{F}(\mathbb{R}, S)$ which belong locally to G, that is*

$$\forall f \in \mathcal{F}(\mathbb{R}, S) \qquad f \in \widetilde{G} \Leftrightarrow \forall s \in S\ \exists f_s \in G\ \forall t \le s \quad f_s(t) = f(t)$$

Thanks to this theorem we immediately identify what is the Cauchy-completion, $\widetilde{\mathbb{R}_\lambda^\Gamma}$, of $\mathbb{R}_\lambda^\Gamma$.

[1] Note that $\mathcal{F}(\mathbb{R}, S)$ is isomorphic to $\mathbb{R}\left(\left(t^S\right)\right)$

Corollary 6.1.16. $\widetilde{\mathbb{R}^\Gamma_\lambda} = \mathbb{R}^\Gamma_\lambda \cup \left\{ \sum_{i<\lambda} r_i\omega^{a_i} \;\middle|\; r_i \in \mathbb{R} \quad a_i \in \Gamma \quad (a_i)_{i<\lambda} \textit{ is decreasing and coinitial with } \Gamma \right\}$

Corollary 6.1.17. *Let $(\Gamma_i)_{i\in I}$ be a family of subgroups of* **No** *such that for all $i, j \in I$ there is some $k \in I$ such that $\Gamma_i \cup \Gamma_j \subseteq \Gamma_k$. Then*

$$\widetilde{\mathbb{R}^{(\Gamma_i)_{i\in I}}_\lambda} = \mathbb{R}^{(\Gamma_i)_{i\in I}}_\lambda \cup \left\{ \sum_{i<\lambda} r_i\omega^{a_i} \;\middle|\; \begin{array}{c} r_i \in \mathbb{R} \\ \forall i < \lambda\ \exists k \in I\ \forall j < i\ a_j \in \Gamma_k\ \ (a_i)_{i<\lambda} \textit{ is decreasing and coinitial with } \bigcup_{i\in I} \Gamma_i \end{array} \right\}$$

6.2 Gap compactness

We now discuss a new notion of compactness. Indeed, the usual expected notion of compact set defined as a closed bounded subset (in finite dimension) or a set that has a finite sub-covering for any covering by open intervals fails in the context of surreal numbers. For instance, $[\,0\,;+\infty\,)$ is closed bounded (by ω for instance) in **No** but we do not want it to be considered as compact. Moreover $\{(\,n-1\,;n\,)\}_{n\in\mathbb{N}}$ is a covering by open intervals of $[\,0\,;+\infty\,)$ but has no finite sub-covering. We then need to be more accurate and introduce the notion of gap-compactness.
In this section, as always, λ is some ε-number and Γ a divisible group.

6.2.1 Interval topology

Definition 6.2.1 (Open interval)**.** An open interval is a set $(\,a\,;b\,)$ with a, b being elements of $\widetilde{\mathbb{R}^\Gamma_\lambda}$ or being non-trivial gaps of $\widetilde{\mathbb{R}^\Gamma_\lambda}$. The interval topology is the topology generated by open interval. A closed interval is an interval of the form $[\,a\,;b\,]$ with $a, b \in \widetilde{\mathbb{R}^\Gamma_\lambda}$.

Remark 6.2.2. A closed bounded interval is an interval of the form $[\,a\,;b\,]$ with $a, b \in \widetilde{\mathbb{R}^\Gamma_\lambda}$. In particular $(\,-\infty\,;+\infty\,)$ is closed in $\widetilde{\mathbb{R}^\Gamma_\lambda}$ and is bounded but we do not want it to be compact.

Remark 6.2.3. The interval topology is totally disconnected: There are gaps everywhere.

6.2.2 Definition and characterization

Definition 6.2.4. We extend the definition of $[\,L \mid R]$. First, if $\langle L \dashv\vdash R\rangle$ is a gap, we may also write

$$[\,L \mid R] = \langle L \dashv\vdash R\rangle$$

When L and R contain gaps, $[\,L \mid R]$ is defined for $L \leq R$ with $L \cap \mathbf{No} < R \cap \mathbf{No}$ by

$$[\,L \mid R] = \left[(L \cap \mathbf{No}) \cup \bigcup_{\langle L' \dashv\vdash R'\rangle \in L} L' \;\middle|\; (R \cap \mathbf{No}) \cup \bigcup_{\langle L' \dashv\vdash R'\rangle \in R} R' \right]$$

Notice that this may be gap itself. In particular, if $\langle L' \dashv\vdash R'\rangle \in L \cap R$ then $[\,L \mid R] = \langle L' \dashv\vdash R'\rangle$. More over if $L \cap R \neq \varnothing$ then it contains a single element which is a gap. Finally, we may also use this notation in the context of $\widetilde{\mathbb{R}^\Gamma_\lambda}$ instead of **No**.

Definition 6.2.5 ((λ, Γ)-gap-compact set)**.** If $\mathcal{X}$ is a set of open intervals of $\widetilde{\mathbb{R}^\Gamma_\lambda}$, let $\mathcal{B}(\mathcal{X})$ the set of the bounds of theses intervals. Now, a subset $X \subseteq \widetilde{\mathbb{R}^\Gamma_\lambda}$ is said (λ, Γ)-**gap-compact** if any covering $\mathcal{X}$ of X by open intervals such that for all non-trivial gap $\langle L \dashv\vdash R\rangle$ such that $L \cup R = \mathcal{B}(\mathcal{X})$, there is $I \in \mathcal{X}$ such that $\inf I \in L$ and $\sup I \in R$ admits a finite sub-covering. Written with a mathematical formula:

$$\forall \langle L \dashv\vdash R\rangle \in \mathcal{G}_\perp\widetilde{\mathbb{R}^\Gamma_\lambda} \quad L \cup R = \mathcal{B}(\mathcal{X}) \quad (\exists I \in \mathcal{X} \quad \inf I \in L \wedge \sup I \in R) \Rightarrow \left(\exists \mathcal{X}' \subseteq \mathcal{X} \quad |\mathcal{X}'| < \infty \wedge X \subseteq \bigcup_{I \in \mathcal{X}'} I \right)$$

Lemma 6.2.6. *Closed bounded intervals of $\widetilde{\mathbb{R}^\Gamma_\lambda}$ are (λ, Γ)-gap-compact.*

Proof. Let $\mathcal{X}$ be a covering of $[\,a\,;b\,]$ by open intervals (where $a, b \in \widetilde{\mathbb{R}^\Gamma_\lambda}$) and satisfying the conditions of Definition 6.2.5. Without loss of generality, we may assume that for any $I \in \mathcal{X}$, $I \cap [\,a\,;b\,] \neq \varnothing$. Let $\mathcal{X}'_0 = \{\,I \in \mathcal{X} \mid a \in I\}$ and $\mathcal{X}_0 = \mathcal{X} \setminus \mathcal{X}'_0$. We define $\mathcal{X}_{n+1}$ and $\mathcal{X}'_{n+1}$ as follows :

- $\mathcal{X}'_{n+1} = \mathcal{X}'_n \cup \{\,I \in \mathcal{X}_n \mid \exists J \in \mathcal{X}'_n \quad I \cup J \text{ is an interval}\}$
- $\mathcal{X}_{n+1} = \mathcal{X} \setminus \mathcal{X}'_{n+1}$

Notice that $(\mathcal{X}'_n)_{n\in\mathbb{N}}$ is an increasing sequence of sets of intervals. Now let

$$\mathcal{C} = \left\{ \inf I, \sup I \;\middle|\; I \in \bigcup_{n\in\mathbb{N}} \mathcal{X}'_n \right\}$$

and

$$\mathcal{D} = \left\{ \inf I, \sup I \;\middle|\; I \in \bigcap_{n\in\mathbb{N}} \mathcal{X}_n \right\}$$

Notice that since $\mathcal{X}$ is a covering hence $a \in I$ for some $I \in \mathcal{X}$ and then $\mathcal{X}_0 \neq \varnothing$. Therefore $\mathcal{C} \neq \varnothing$. We also have $\mathcal{C} \cup \mathcal{D} = \mathcal{B}(\mathcal{X})$. Note that by induction one can show that $\mathcal{C} \leq \mathcal{D}$.
Assume $\mathcal{C} \cap \mathcal{D} \neq \varnothing$. Then it contain exactly one element, x. If $x \in [\,a\,;b\,]$ then there is $I \in \mathcal{X}$ such that $x \in I$. Let $I' \in \bigcup_{n\in\mathbb{N}} \mathcal{X}'_n$ such that $x = \sup I'$. Then $I \cap I' \neq \varnothing$ and then $I \in \bigcup_{n\in\mathbb{N}} \mathcal{X}'_n$. Then, $\sup I \in \mathcal{C}$ and this contradicts $\mathcal{C} \leq \mathcal{D}$.
Therefore x is a gap. But again there is $I \in \mathcal{C}$ and $J \in \mathcal{D}$ such that $\sup I = \inf J = x$ and therefore $J \in \bigcup_{n\in\mathbb{N}} \mathcal{X}'_n$, hence $\sup J \in \mathcal{C}$ which again contradicts $\mathcal{C} \leq \mathcal{D}$. That leads to $\mathcal{C} \cap \mathcal{D} = \varnothing$.
Assume $\mathcal{D} \neq \varnothing$. By the previous paragraph, $\mathcal{C} < \mathcal{D}$. Applying the definition of $\mathcal{X}$ to $\mathcal{C}$ and $\mathcal{D}$, we have some $I \in \mathcal{X}$ such that $\inf I \in \mathcal{C}$ and $\sup I \in \mathcal{D}$. Since $\bigcup_{n\in\mathbb{N}} \mathcal{X}'_n$ and $\bigcap_{n\in\mathbb{N}} \mathcal{X}_n$ form a partition of $\mathcal{X}$, I must belong to one of them. These leads to $\mathcal{C} \cap \mathcal{D} \cap \{\inf I, \sup I\} \neq \varnothing$ which is again a contradiction. Therefore $\mathcal{D} = \varnothing$.
What precedes show that $\mathcal{X} = \bigcup_{n\in\mathbb{N}} \mathcal{X}'_n$. Let $I_0 \in \mathcal{X}$ such that $b \in I_0$. Let n such that $I_0 \in \mathcal{X}_n$. Assume we have constructed $I_k \in \mathcal{X}'_{n-k}$ for $k < n$. There is some $I_{k+1} \in \mathcal{X}'_{n-k-1}$ such that $I_{k+1} \cup I_k$ is an interval. In particular $\bigcup_{k=0}^{n} I_k$ is an interval, $a \in I_k$ and $b \in I_0$. Hence, $\{\, I_k \mid k \in [\![\, 0\,;\,n\,]\!] \}$ is a finite sub-covering of $\mathcal{X}$.

□

We can do better and give a characterization of what a compact is, that is analogous to the real case.

Definition 6.2.7 (Gap-connected set)**.** $X \subseteq \widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$ is said to be **gap-connected** if for any non-trivial gap $\langle L \dashv\vdash R \rangle$ such that $L \subseteq X$ or $R \subseteq X$, there are $L', R' \subseteq X$ such that $\langle L \dashv\vdash R \rangle = \langle L' \dashv\vdash R' \rangle$.

That is to say, a gap-connected set must approach its gaps by both sides.

Example 6.2.8.

- An interval with surreal bounds is gap-connected.
- $X = [\,0\,; {+\infty}\,) \cup [\,\omega\,; \omega+1\,]$ is not gap-connected. For instance ${+\infty} = \langle L \dashv\vdash R \rangle$ with $L \subseteq X$ but we cannot find $R' \subseteq X$ such that $\langle L \dashv\vdash R \rangle = \langle L \dashv\vdash R' \rangle$.
- $[\,1\,; {+\infty}\,)$, despite being bounded (in $\widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$) and closed (in the sense of the topology), is not gap-connected.

Lemma 6.2.9. *Let $X \subseteq \widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$ be bounded, closed and gap-connected. Then X has a minimal element and a maximal element.*

Proof. Assume X has no minimal element. Let $(x_i)_i$ be a decreasing sequence of X coinitial with X. This sequence must have a limit ordinal length ν otherwise X has a minimal element. This sequence has no limit, otherwise X would not be closed. Since X is bounded, there is L such that we have a gap $\langle L \dashv\vdash \{\, x_i \mid i < \nu \}\rangle$. Since $(x_i)_{i<\nu}$ has no limit, this can't be a trivial gap. X is gap-connected. Therefore, there is $L' \subseteq X$ such that

$$\langle L' \dashv\vdash \{\, x_i \mid i < \nu \}\rangle = \langle L \dashv\vdash \{\, x_i \mid i < \nu \}\rangle$$

which contradicts the coinitiality property. We do the same proof for the existence of a maximal element. □

Proposition 6.2.10. *$X \subseteq \widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$ is (λ, Γ)-gap-compact if and only if X is bounded closed and gap-connected.*

Proof. Let α be the cofinality of $\widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$.

(NC ⇒) Let X be (λ, Γ)-gap-compact. Assume that there is no upper bound on X. Then there is an increasing sequence $(x_i)_{i<\alpha}$ of X that is cofinal with $\widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$. Without loss of generality, we may assume that this sequence is increasing and that there is some $\varepsilon \in \left(\widetilde{\mathbb{R}^{\Gamma}_{\lambda}}\right)^*_+$ such that for $i \neq j$ $|x_i - x_j| > \varepsilon$. Indeed, we can extract such a sequence and if we get something shorter, it contradicts the minimality of α. Set $x_{-1} = \mathbf{Off}$. Now for every ordinal $i < \alpha$, set

- ➢ $I_i = (\,x_{i-1}\,; x_{i+1}\,)$ if i is a successor ordinal
- ➢ $I_i = (\,x_j\,; x_{i+1}\,)$, where $j < i$ chosen arbitrarily, otherwise (for $i = 0$, the only possible choice is $j = -1$).

Now, let $\mathcal{X} = \{I_i\}_{i<\alpha}$. It is a covering of X by open intervals. Since $(x_i)_{i<\alpha}$ is increasing, it is well ordered, and so is $\mathcal{B}(\mathcal{X}) \subseteq \{x_i\}_{i<\alpha}$. Let $L < R$ such that $L \cup R = \mathcal{B}(\mathcal{X})$. Then R has a lowest element, say x_i. In this case I_i is such that $\sup I_i \in R$ and $\inf I \in L$. By definition of gap-compactness, we can extract a finite subcovering $\mathcal{X}'$ of $\mathcal{X}$. One one hand we have

$$\max\{\sup I \mid I \in \mathcal{X}'\} \notin \bigcup_{I\in\mathcal{X}'} I \supseteq X$$

and on the other hand
$$\max\{\sup I \mid I \in \mathcal{X}'\} \in \{x_i\}_{i<\alpha} \subseteq X$$

which is a contradiction. Then there is an upper bound for X. A similar proof shows that there is a lower bound of X. That means that X is bounded. We now show that it is closed. Assume the opposite. Let $(x_i)_{i<\alpha}$ be a sequence of X that converges to $x \notin X$. Since α is also the coinitiality of $\left(\widetilde{\mathbb{R}^{\Gamma}_{\lambda}}\right)^*_+$, we assume that the sequence is monotonic. For instance, let us assume it is increasing. Define again the I_i as before and if $I_i = (\,a\,;b\,)$ define $J_i = (\,2x-b\,;2x-a\,)$ and consider the covering

$$\mathcal{X} = \{I_i\}_{i<\alpha} \cup \{J_i\}_{i<\alpha}$$

Again, it must have a finite sub-covering which is impossible. Then it is closed. Finally, we have to prove that it is gap-connected. Assume it is not and take a gap $\langle L \dashv\vdash R\rangle$ such that $L \subseteq X$ or $R \subseteq X$ and such there there is no $L', R' \subseteq X$ such that $\langle L \dashv\vdash R\rangle = \langle L' \dashv\vdash R'\rangle$. For instance say $L \subseteq X$. Then, there is an element $r \in (\,\langle L \dashv\vdash R\rangle\,;\mathbf{On}\,)$ such that for all $x \in X \cap (\,\langle L \dashv\vdash R\rangle\,;\mathbf{On}\,)$, $r \leq x$. Take $(x_i)_i$ increasing and cofinal with L. This sequence has a limit ordinal length (other wise $\langle L \dashv\vdash R\rangle$ would be a trivial gap). Consider the following covering of X:

$$\mathcal{X} = \{(\,\mathbf{Off}\,;x_i\,)\}_i \cup \{(\,r\,;\mathbf{On}\,)\}$$

Again, this must accept some finite sub-covering which is impossible.

($\overset{SC}{\Leftarrow}$) Let X be bounded closed and gap-connected. Applying Lemma 6.2.9, X has both a minimal element a and a maximal element b. Let $\mathcal{X}$ be a covering of X by open intervals satisfying the condition of Definition 6.2.5. For $k, n \in \mathbb{N}$, we define a_k and $\mathcal{Y}_{k,n}$ as follows:

- ➢ $a_0 = a$
- ➢ $\mathcal{Y}_{k,0} = \{I \in \mathcal{X} \mid a_k \in I\}$
- ➢ $\mathcal{Y}_{k,n+1} = \mathcal{Y}_{k,n} \cup \{I \in \mathcal{X}_n \mid \exists J \in \mathcal{Y}_{k,n} \quad I \cup J \text{ is an interval}\}$
- ➢ $$a_{k+1} = \begin{cases} \min\left(X \setminus \bigcup_{p\leq k, n\in\mathbb{N}} \bigcup_{I\in\mathcal{Y}_{p,n}} I\right) & \text{if } X \neq \bigcup_{p\leq k, n\in\mathbb{N}} \bigcup_{I\in\mathcal{Y}_{p,n}} I \\ \mathbf{On} & \text{otherwise} \end{cases}$$

Assume that this is not well defined, and that k is minimum such that a_{k+1} is not defined (this is the only case where the definition can fail). Let $X' = \bigcup_{p\leq k, n\in\mathbb{N}} \bigcup_{I\in\mathcal{Y}_{p,n}} I$. We then have that $X' \neq X$ and X' has no minimum. Let $(x_i)_{i<\nu}$ be a decreasing coinitial sequence of X'. ν must be a limit ordinal.

- ➢ If $(x_i)_{i<\nu}$ converges to x, $x \in X$. By coinitiality, $x \notin X'$. Then there is $I \in \mathcal{Y}_{k,0} \cup \bigcup_{p\leq k, n\in\mathbb{N}} \mathcal{Y}_{p,n}$ such that $x \in I$. Since I is open, for i large enough, $x_i \in I$. This is a contradiction with the definition of X'.
- ➢ If $(x_i)_{i<\nu}$ does not converges. Then there is $L \subseteq \left(X \cap \bigcup_{p\leq k, n\in\mathbb{N}} \bigcup_{I\in\mathcal{Y}_{p,n}} I\right)$ such that we have a gap $G := \langle L \dashv\vdash \{x_i \mid i < \nu\}\rangle$. G cannot be a trivial gap, otherwise L has a maximum and $(x_i)_{i<\nu}$ converges to it. By definition of $\mathcal{X}$, there is $I \in \mathcal{X}$ such that $\inf I \leq l$ for some $l \in L$ and $\sup I \geq x_i$ for some i. Therefore there is $J \in \bigcup_{p\leq k, n\in\mathbb{N}} \mathcal{Y}_{p,n}$ such that $I \cup J$ is an interval. For instance $J \in \mathcal{Y}_{p,n}$. Then $I \in \mathcal{Y}_{p,n+1}$ which is a contradiction.

By a similar argument we can define $b_k = \max\left(X \cap \bigcup_{p\leq k, n\in\mathbb{N}} \bigcup_{I\in\mathcal{Y}_{p,n}} I\right)$ for $k \in \mathbb{N}$. We have $a_k \leq b_k \leq a_{k+1}$ and if $a_k \neq \mathbf{On}$ then $b_k < a_{k+1}$.

Now that we know that everything is well defined, we claim that there is $k \in \mathbb{N}$ such that $a_k = \mathbf{On}$. Assume this is not true. Therefore $(a_k)_{k\in\mathbb{N}}$ is an increasing sequence of X bounded by b. With the same argument as above, $(a_k)_{k\in\mathbb{N}}$ does not converges and then there is $R \subseteq X$ such that we have a gap $\langle\{a_k \mid k \in \mathbb{N}\} \dashv\vdash R\rangle$. With the same argument as above, this is again impossible. Therefore there is a minimum $k \in \mathbb{N}$ such that $a_k = \textbf{On}$ and then

$$X \subseteq \bigcup_{p<k, n\in\mathbb{N}} \bigcup_{I\in\mathcal{Y}_{p,n}} I$$

Notice that $\bigcup_{n\in\mathbb{N}} \mathcal{Y}_{p,n}$ is a covering of $[\, a_p \,;\, b_p \,]$ satisfying the condition of Definition 6.2.5. By Lemma 6.2.6, we can extract a finite subcovering $\mathcal{Y}_p$. Finally $\bigcup_{p<k} \mathcal{Y}_p$ is a finite subcovering of $\mathcal{X}$ that covers X.

□

6.2.3 Covering by open sets

In definition 6.2.5 we specified the covering to be a covering by open intervals. Since open sets are union of intervals one may think that it is easy to deal with covering by open sets. Nevertheless, it is nothing but an illusion because of the suitable generalization of $\mathcal{B}(\mathcal{X})$ we have to find. Because of the metric-like topology, an open set is an union made of open interval. We try to be even more precise so that we will be able to characterize $\mathcal{B}(\mathcal{X})$.

Definition 6.2.11. Two open intervals I and J of $\widetilde{\mathbb{R}^\Gamma_\lambda}$ are said to be **strongly disjoint** if $I \cup J$ is not an interval.

Example 6.2.12. $(\, 0 \,;\, 1 \,)$ and $(\, 1 \,;\, 2 \,)$ are strongly disjoint but not $(\, 0 \,;\, {+\infty} \,)$ and $(\, {+\infty} \,;\, \omega \,)$.

We now pretend that we can express any open set in $\widetilde{\mathbb{R}^\Gamma_\lambda}$ as a union of strongly disjoint intervals.

Lemma 6.2.13. *Let U be an open set in $\widetilde{\mathbb{R}^\Gamma_\lambda}$. It can be written as a union of strongly disjoint intervals.*

Proof. Take U expressed as the union of intervals $U = \bigcup_{I\in\mathcal{I}} I$. On all intervals of $\widetilde{\mathbb{R}^\Gamma_\lambda}$ define the relation $I * J$ meaning that I and J are not strongly disjoint. Take $\star$ to be its transitive closure. $\star$ is an equivalence relation. Define $\mathcal{I}_0 = \mathcal{I}$ and

- If $\mathcal{I}_\alpha$ is defined then $\mathcal{I}_{\alpha+1}$ is as follows. By definition, for any class $C \in \mathcal{I}_\alpha/\star$, C is such that $\bigcup_{I\in C} I$ is an interval denoted I_C. Set
$$\mathcal{I}_{\alpha+1} = \{\, I_C \mid C \in \mathcal{I}_\alpha/\star \}$$
- If α is a limit ordinal and $\mathcal{I}_\beta$ as been defined for $\beta < \alpha$, we build $\mathcal{I}_\alpha$ as follows. First, we notice that if $I \in \mathcal{I}_\beta$ and $\gamma > \beta$ then there is $J \in \mathcal{I}_\gamma$ such that $I \subseteq J$. From that, we build for each $I \in \mathcal{I}$ an increasing sequence $\left(I^{(\beta)}\right)_{\beta<\alpha}$ of interval such that $I^{(\beta)} \in \mathcal{I}_\beta$. Then set
$$I_\alpha = \left\{ \bigcup_{\beta<\alpha} I^{(\beta)} \;\middle|\; I \in \mathcal{I} \right\}$$

By an argument of cardinality, this construction must reach a fixed point at some point. Since it is a fixed point, each interval must be strongly disjoint from the others. Moreover, at each step, the union is not changed, it remains U. Then the fixed point is a writing of U as a union of strongly disjoint open intervals. □

Definition 6.2.14 (Canonical interval representation)**.** Let U be an open set of $\widetilde{\mathbb{R}^\Gamma_\lambda}$. For $x \in U$ we set I_x to be the largest interval of $\widetilde{\mathbb{R}^\Gamma_\lambda}$ containing x and included in U. Then $U = \bigcup_{x\in U} I_x$. The **canonical interval representation** of U is the family of strongly disjoint intervals obtained by the process in the proof of Lemma 6.2.13.

Definition 6.2.15. Let $\mathcal{X}$ be a set of open sets. Let $\mathcal{X}'$ be the set of open interval
$$\mathcal{X}' = \{\, I \mid \exists U \in \mathcal{X} \quad I \in \mathcal{I}_U \}$$
where $\mathcal{I}_U$ is the canonical interval representation of U. Then we set $\mathcal{B}_{set}(\mathcal{X}) = \mathcal{B}(\mathcal{X}')$.

We are now ready to characterize compact sets with coverings by open sets, and not only by open intervals.

Proposition 6.2.16. *X is (λ, Γ)-gap-compact if an only if from any covering $\mathcal{X}$ of X by open set such that for any non-trivial gap $\langle L \dashv\vdash R \rangle$ such that $L \cup R = \mathcal{B}_{set}(\mathcal{X})$, there is $U \in \mathcal{X}$ that is a neighborhood of $\langle L \dashv\vdash R \rangle$, we can extract a finite subcovering.*

Proof. (SC $\Leftarrow$) Take $\mathcal{X}$ be a covering by open interval satisfying Definition 6.2.5. We clearly have $\mathcal{B}_{set}(\mathcal{X}) \subseteq \mathcal{B}(\mathcal{X})$. Let $\langle L \dashv\vdash R \rangle$ be a non-trivial gap such that $L \cup R = \mathcal{B}(\mathcal{X})$. Then either $\mathcal{B}(\mathcal{X}) \setminus \mathcal{B}_{set}(\mathcal{X}) < \langle L \dashv\vdash R \rangle$ or $\mathcal{B}(\mathcal{X}) \setminus \mathcal{B}_{set}(\mathcal{X}) > \langle L \dashv\vdash R \rangle$. In both cases we can get $L' \subseteq L$ and $R' \subseteq R$ such that $\langle L \dashv\vdash R \rangle = \langle L' \dashv\vdash R' \rangle$ and $L' \cup R' = \mathcal{B}_{set}(\mathcal{X})$. Then there is $I \in \mathcal{X}$ such that I is a neighborhood of $\langle L' \dashv\vdash R' \rangle$, that is $\inf I \in L' \subset L$ and $\sup I \in R' \subset R$.

Take $\mathcal{X}$ be a covering of X by open sets. Let $\mathcal{X}'$ obtained as in the previous definition. $\mathcal{X}'$ is covering by open interval satisfying the conditions of Definition 6.2.5. Then we can extract a finite subcovering. Let us call it $\mathcal{Y}'$. For $I \in \mathcal{Y}'$ take $U_I \in \mathcal{X}$ such that $I \subseteq U_I$. It is possible by construction. Then

$$\mathcal{Y} := \{U_I \mid I \in \mathcal{Y}'\}$$

is a finite subcovering of $\mathcal{X}$.

□

6.3 Gap continuity

6.3.1 Definitions

We now introduce the notion of gap-continuous functions. In fact we want functions to be continuous even on gaps. We then give a notion of neighborhood for a gap $\langle L \dashv\vdash R\rangle$.

Definition 6.3.1. An open interval I with bounds in $\widetilde{\mathbb{R}^\Gamma_\lambda}$ is a neighborhood of the gap $\langle L \dashv\vdash R\rangle$ if there are $l \in L$ and $r \in R$ such that

$$\inf I < l < r < \sup I$$

Definition 6.3.2. A function $f : \widetilde{\mathbb{R}^\Gamma_\lambda} \to \widetilde{\mathbb{R}^\Gamma_\lambda}$ is said to be (λ, Γ)-**gap-continuous** if it satisfies:

GC1. f is continuous (*i.e* satisfies Definition 6.0.1).

GC2. For any non-trivial gap $G = \langle L \dashv\vdash R\rangle \in \mathcal{G}_\perp\widetilde{\mathbb{R}^\Gamma_\lambda}$, there is some $y \in \widetilde{\mathbb{R}^\Gamma_\lambda} \cup \mathcal{G}_\perp\widetilde{\mathbb{R}^\Gamma_\lambda}$ such that for any neighborhood J of y, there is some neighborhood I of G such that

$$x \in I \implies f(x) \in J$$

Moreover, if $y \in \widetilde{\mathbb{R}^\Gamma_\lambda}$ then we must have $y \in f(I)$.

GC3. For any non-trivial gap $G = \langle L \dashv\vdash R\rangle \in \mathcal{G}_\perp\widetilde{\mathbb{R}^\Gamma_\lambda}$, if y given by **GC2.** is a gap, for any neighborhood I of G

$$f(I) \cap \left\{ z \in \widetilde{\mathbb{R}^\Gamma_\lambda} \;\middle|\; z > y \right\} \neq \varnothing \qquad \text{and} \qquad f(I) \cap \left\{ z \in \widetilde{\mathbb{R}^\Gamma_\lambda} \;\middle|\; z < y \right\} \neq \varnothing$$

f is (λ, Γ)-**weakly-gap-continuous** if it satisfies **GC1.** and **GC2.**.

Remark 6.3.3. This is a quite intuitive extension of the definition of a continuous function. Nevertheless, one can notice that we require that gap-continuous functions must "show gaps around the gaps". That's basically because we would like to have the Intermediate Value Theorem and the Extreme Value Theorem.

Remark 6.3.4. Instead of taking f defined on all $\widetilde{\mathbb{R}^\Gamma_\lambda}$, we can consider f being defined on a subinterval I of it. In that case, the gap G in Axioms **GC2.** and **GC3.** must be taken such that I is an neighborhood of G.

Example 6.3.5. To put some visual aspects on Definition 6.3.2, we can have a look to the following figure:

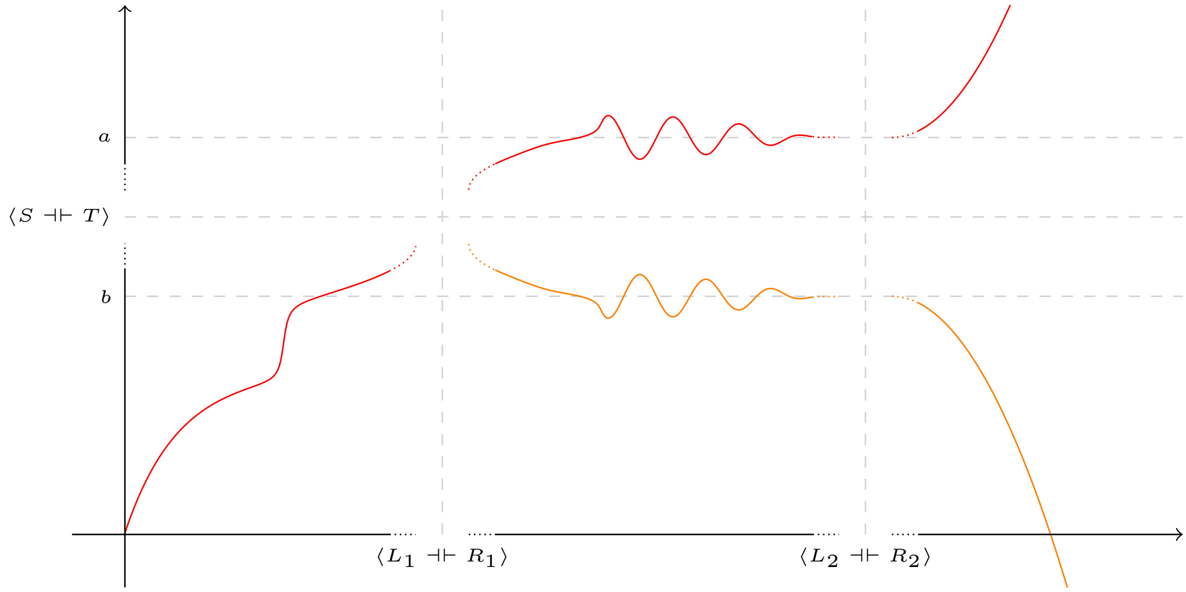


Figure 6.1: The difference between gap-continuity and weak gap-continuity.

In the above figure, the red function is gap-continuous because both sides of $\langle S \dashv\vdash T \rangle$ are touched around $\langle L_1 \dashv\vdash R_1 \rangle$, which is not true if we consider the orange part on the right. The function consisting in the red part on the left and the the orange part is only weakly-gap-continuous.

Example 6.3.6. The function $\exp$ is gap-continuous on $\widetilde{\mathbb{R}_\lambda^\Gamma}$ and $\ln$ is gap-continuous on $\widetilde{\mathbb{R}_\lambda^\Gamma}^*$.

Example 6.3.7. The derivation ∂ is not weakly-gap-continuous. On any gap of the form $G = x \pm \frac{1}{+\infty}$ for $x = \sum_{i<\nu} r_i \omega^{a_i}$ where $a_i \geq 0$ for $i < \nu$, there is no neighborhood of G such that ∂ stays close to some gap or some surreal number. In fact, the derivation ∂ behave somehow like the figure below around G:

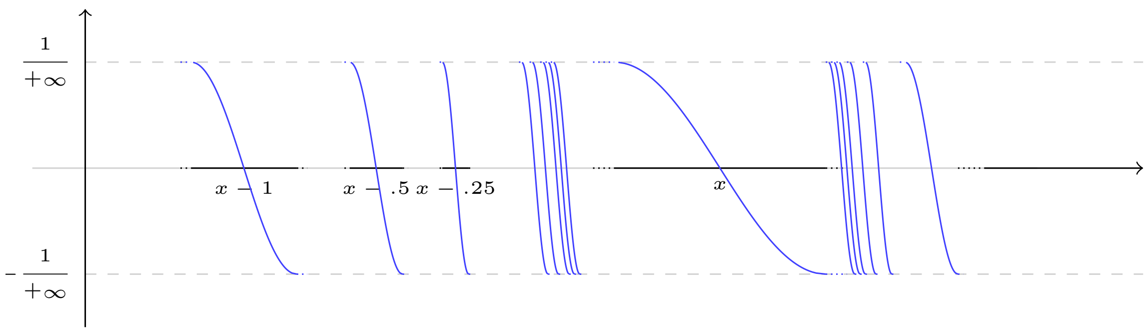


Figure 6.2: Weak-gap-continuity failure of ∂.

In fact, the curve on the right side of the gap $x - \frac{1}{+\infty}$ repeats itself infinitely many times on the left side. This is because there is one curve for each $x - \varepsilon$ for all real number $\varepsilon > 0$. For the same reason, a similar phenomenon occurs at the gap $x + \frac{1}{+\infty}$.

In the next paragraph we show that weak-gap-continuity is sufficient for the Intermediate Value Theorem. Note that even if ∂ is not weakly-gap-continuous, it may still satisfy some form of the Intermediate Value Theorem. The gaps where it fails are very special and don't seem to break the theorem. Moreover, in the case of transseries, it is already known that the derivation satisfies the Intermediate Value Theorem (see [51, 2, 3])

6.3.2 Intermediate value theorem

The very first thing we can notice is that, unless general continuous functions, (λ, Γ)-gap-continuous functions have sufficient property to satisfy the Intermediate Value Theorem. Obviously, we do not have the other direction, but it is also the case in real analysis.

Theorem 6.3.8 (Intermediate value theorem). *Let $f : \widetilde{\mathbb{R}_\lambda^\Gamma} \to \widetilde{\mathbb{R}_\lambda^\Gamma}$ be (λ,Γ)-weakly-gap-continuous and $a < b$ such that assume $f(a) \leq f(b)$. Then, for all $y \in [\min(f(a), f(b)) ; \max(f(a), f(b))]$ there is $a \leq c \leq b$ such that $f(c) = y$.*

Proof. Assume for instance $f(a) \leq f(b)$. Let $y \in [f(a) ; f(b)]$ and A defined as follows

$$A = \{a' \mid \forall a'' \in [a ; a''] \quad f(a'') \leq y\}$$

By definition, A is an interval. We have two subcases :

- $\sup A \in \widetilde{\mathbb{R}_\lambda^\Gamma}$. Then, using continuity of f, we have $\sup A \in A$. Again, using continuity, if $f(\sup A) < y$, then there is $\varepsilon > 0$ such that $f(\sup A + \varepsilon') < y$ for all $0 \leq \varepsilon' < \varepsilon$ what is impossible. Then $\sup A$ is the desired c.
- $\sup A \notin \widetilde{\mathbb{R}_\lambda^\Gamma}$. f is weakly-gap-continuous. Let z given by Axiom **GC2.** around the gap $\sup A$.
  - ➢ If $z \in \widetilde{\mathbb{R}_\lambda^\Gamma}$ then it must be reached on any neighborhood of $\sup A$. If $z > y$ the continuity gives a contradiction with the definition of A. If $z < y$, the continuity give a contradiction with the definition of $\operatorname{supp} A$. Therefore, $y = z$. Take any neighborhood I of $\operatorname{supp} A$ such that $a, b \notin I$, that is $a < I < b$. Hence, there is some $c \in I$ such that $f(c) = y$ that is what we expected.
  - ➢ If $z \notin \widetilde{\mathbb{R}_\lambda^\Gamma}$, *i.e* z is a gap, we have either $z < y$ or $z > y$ what is impossible by the same argument as in the previous case.

□

6.3.3 Extreme value theorem

In the context of real numbers, continuous functions satisfy the extreme value theorem. More precisely, the image of a compact by a continuous function is also compact. We provide a counterpart in the context of surreal numbers.

Theorem 6.3.9. *Let $f : \widetilde{\mathbb{R}_\lambda^\Gamma} \to \widetilde{\mathbb{R}_\lambda^\Gamma}$ be a (λ,Γ)-gap-continuous function. Let $X \subseteq \widetilde{\mathbb{R}_\lambda^\Gamma}$ be a (λ,Γ)-gap-compact set, then $f(X)$ is also (λ,Γ)-gap-compact .*

Proof. We will prove that $f(X)$ is bounded, closed and gap-connected.

- Assume $f(X)$ is not bounded. Consider a sequence of $f(X)$, $(y_i)_{i<\alpha}$, where α is the cofinality of $\widetilde{\mathbb{R}_\lambda^\Gamma}$, increasing and cofinal with $\widetilde{\mathbb{R}_\lambda^\Gamma}$. Denote $x_i \in X$ an element such that $f(x_i) = y_i$. We can extract a monotonic sequence from $(x_i)_{i<\alpha}$. Up to consider the function $x \mapsto f(-x)$, we can assume that $(x_i)_{i<\alpha}$ is increasing. If it converges to x, the continuity of f ensures that $(y_i)_{i<\alpha}$ converges to $f(x)$ what is not true. Therefore, there is $R \subseteq X$ such that we have a non-trivial gap $G := \langle \{x_i \mid i < \alpha\} \dashv\vdash R\rangle$. Let y given by **GC2.** for G. If $y \in \widetilde{\mathbb{R}_\lambda^\Gamma}$, then for sufficiently large i, x_i is arbitrarily close to y, which is not true. Therefore y is gap. Since $(y_i)_{i<\alpha}$ is increasing and cofinal with $\widetilde{\mathbb{R}_\lambda^\Gamma}$, we then have $y = \mathbf{On}$. By Axiom **GC3.**, there must be some $x \in X$ such that $f(x) > \mathbf{On}$ which is a contradiction. Therefore $f(X)$ is bounded.
- Take a sequence $(y_i)_{i<\alpha}$ monotonic and that converge to y. Let x_i such that $f(x_i) = y_i$. Again we can extract a monotonic sequence from x_i. If $(x_i)_{i<\alpha}$ converges to x, then $x \in X$ and $f(x) = y$ which is a contradiction. Assume for instance that both sequences are increasing. There is a gap $\langle \{x_i \mid i < \alpha\} \dashv\vdash R\rangle$. Applying then Axiom **GC2.** we get z. Clearly $z = y$. Again, **GC2.** ensure that $y \in f(X)$. $f(X)$ is closed.
- Let $\langle L \dashv\vdash R\rangle$ a non-trivial gap such that $L \subseteq f(X)$. We again build a monotonic, for instance increasing, sequence $(x_i)_{i<\alpha}$ such that $(f(x_i))_{i\in\alpha}$ is a sequence of L cofinal with L. Because of continuity, $(x_i)_{i<\alpha}$ cannot converge. Since X is gap-compact, there is some $R' \subseteq X$ such that we have a gap $\langle \{x_i \mid u < \alpha\} \dashv\vdash R\rangle$. Axioms **GC2.** and **GC3.** ensure that we can find $R'' \subseteq f(X)$ such that $\langle L \dashv\vdash R\rangle = \langle L \dashv\vdash R''\rangle$. $f(X)$ is gap-connected.

By Proposition 6.2.10, $f(X)$ is gap-compact.

□

Theorem 6.3.10 (Extreme values theorem). *Let $f : \widetilde{\mathbb{R}_\lambda^\Gamma} \to \widetilde{\mathbb{R}_\lambda^\Gamma}$ be a (λ,Γ)-gap-continuous function. Let $X \subseteq \widetilde{\mathbb{R}_\lambda^\Gamma}$ be (λ,Γ)-gap-compact . Then f reaches its extrema on X.*

Proof. Using Theorem 6.3.9, we know that $f(X)$ is (λ,Γ)-gap-compact . Using Proposition 6.2.10, we know that it is bounded, closed and gap-connected. By Lemma 6.2.9 it has a maximum and a minimum. □

Remark 6.3.11. Gap-continuous function satisfy the intermediate value theorem and the extreme value theorem. These theorems are fine but are quite difficult to apply when we are interested in multiple function at the same time. Basically, if f and g are (λ,Γ)-gap-continuous , $f - g$ may not, and then could violate the Intermediate Value Theorem, or the Extreme Value Theorem what is unsuitable. For instance, the function $f : x \mapsto x$ and $g : x \mapsto \begin{cases} x & \text{if } x < \infty \\ x - 1 & \text{otherwise} \end{cases}$ are (λ,Γ)-gap-continuous but $f - g$ is not. In particular, Rolle's theorem cannot be proven that way.

Chapter 7

Oscillating numbers

This chapter is an attempt to formalize oscillating numbers. This numbers are our suggestion to handle the oscillations in the context of surreal numbers. Indeed, surreal number cannot handle osculating behavior. For instance, there is no surreal number such that $\partial\partial x = -x$. In particular, $\cos\omega$ makes no sense in the context of surreal numbers. We then want to have new numbers so that we can give a sense to the expression $\cos\omega$, but we try to make it as simple as possible and use structures that makes sense. Namely, we try to use a structure that is no just about the introduction of a new symbol for $\cos x$ when $x \succ 1$.

In this PhD thesis, van der Hoeven tackled a very similar problem in the context of transseries (see [49, Sections 4.6, 6.7]). In particular, he introduced complex transseries to deal with the oscillations. Moreover, his approach solves algebraic differential equations (see [50]). This fact gives hope to solve such equations with oscillating numbers.

As said above, this small chapter contains few contributions. We are not fully happy with its current state but we believe it opens some very interesting concepts and statements. We provide the definition of a new structure to give a direction for future work to be done, which will be discussed in Chapter 8.

- Section 7.1 introduces the definition of an oscillating number.
- Section 7.2 provide the osculating numbers a structure of ring and proves that all the operations are well defined.

This chapter contains the following original contribution:

- The formalization of our proposed concept of oscillating numbers (Definition 7.1.1).
- The proof that oscillating numbers support ring operations (Propositions 7.2.3 and 7.2.4).

7.1 Definition

Definition 7.1.1. Let $\mathbb{K}$ a field of surreal numbers that is stable under $\exp$ and $\ln$. Let $\mathbb{K}_\infty$ the set (or class) of the purely infinite numbers in $\mathbb{K}$. Notice that $\mathbb{K}$ must contain $\mathbb{Q}$ and therefore, $\mathbb{K}_\infty$ is a divisible group. Let $\mathbb{K}_\infty^+$ the set of non-negative elements in $\mathbb{K}_\infty$. We introduce the set (or class) of **oscillating numbers** over $\mathbb{K}$, the set (or class) defined as follows:

$$\mathbb{O}_\mathbb{K} = \left\{ \sum_{x\in\mathbb{K}_\infty} \exp(x)(C(\varphi_x) + S(\psi_x)) \;\middle|\; \begin{array}{c} \varphi_x : \mathbb{K}_\infty^+ \to \mathbb{R} \quad \text{and} \quad \psi_x : \mathbb{K}_\infty^+ \to \mathbb{R}, \psi_x(0) = 0 \\ (\varphi_x(z))_{z\in\mathbb{K}_\infty^+} \text{ and } (\psi_x(z))_{z\in\mathbb{K}_\infty^+} \text{ are summable} \\ \{\, x \in \mathbb{K}_\infty \mid \operatorname{supp}\varphi_x \cup \operatorname{supp}\psi_x \neq \varnothing \} \text{ is reverse well-ordered} \\ \operatorname{supp}\varphi_x \cup \operatorname{supp}\psi_x \text{ is contained in a free } \mathbb{Z}\text{-module of finite dimension} \end{array} \right\}$$

The intuition behind this writing is that the function φ_x and ψ_x give to each "frequency" some coefficient. The element $C(\varphi_x) + S(\psi_x)$ may look like a "Fourier expansion" of some "function". More precisely, we will get inspired from the Fourier series to define these series. $C(\varphi_x)$ will stand for the cosine part and $S(\psi_x)$ for the sine part.

Remark 7.1.2. The expression $\sum\limits_{x\in\mathbb{K}_\infty} \exp(x)(C(\varphi_x) + S(\psi_x))$ is just a writing to give an intuition of what is going on. A more formal definition would be

$$\mathbb{O}_\mathbb{K} = \left\{ (\varphi_x, \psi_x)_{x\in\mathbb{K}_\infty} \;\middle|\; \begin{array}{c} \varphi_x : \mathbb{K}_\infty^+ \to \mathbb{R} \quad \text{and} \quad \psi_x : \mathbb{K}_\infty^+ \to \mathbb{R}, \psi_x(0) = 0 \\ (\varphi_x(z))_{z\in\mathbb{K}_\infty^+} \text{ and } (\psi_x(z))_{z\in\mathbb{K}_\infty^+} \text{ are summable} \\ \{\, x \in \mathbb{K}_\infty \mid \operatorname{supp}\varphi_x \cup \operatorname{supp}\psi_x \neq \varnothing \} \text{ is reverse well-ordered} \\ \operatorname{supp}\varphi_x \cup \operatorname{supp}\psi_x \text{ is contained in a free } \mathbb{Z}\text{-module of finite dimension} \end{array} \right\}$$

The problem with such a notation is that we would hardly understand why the multiplication over oscillating numbers is defined as it is in the following.

7.2 Operations

Definition 7.2.1. Consider two oscillating numbers follows:

$$a = \sum_{x \in \mathbb{K}_\infty} \exp(x)(C(\varphi_x) + S(\psi_x)) \qquad \text{and} \qquad b = \sum_{x \in \mathbb{K}_\infty} \exp(x)(C(\varphi'_x) + S(\psi'_x))$$

Motivated by the above intuition, we define the addition and multiplication operations as follows:

$$a + b = \sum_{x \in \mathbb{K}_\infty} \exp(x)(C(\chi_x) + S(\chi'_x))$$

where

$$\chi_x : \left\{ \begin{array}{ccc} \mathbb{K}_\infty^+ & \to & \mathbb{R} \\ z & \mapsto & \varphi_x(z) + \varphi'_x(z) \end{array} \right. \quad \text{and} \quad \chi'_x : \left\{ \begin{array}{ccc} \mathbb{K}_\infty^+ & \to & \mathbb{R} \\ z & \mapsto & \psi_x(z) + \psi'_x(z) \end{array} \right.$$

$$ab = \sum_{x \in \mathbb{K}_\infty} \exp(x)(C(\chi_x) + S(\chi'_x))$$

where

$$\chi_x : \left\{ \begin{array}{ccc} \mathbb{K}_\infty^+ & \to & \mathbb{R} \\ z & \mapsto & \frac{1}{2} \sum\limits_{\substack{z = z_1 + z_2 \\ x_1 + x_2 = x}} \left(\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2) - \psi_{x_1}(z_1)\psi'_{x_2}(z_2) \right) \\ & & + \frac{1}{2} \sum\limits_{\substack{z = |z_1 - z_2| \\ x_1 + x_2 = x}} \left(\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2) \right) \end{array} \right.$$

and

$$\chi'_x : \left\{ \begin{array}{ccc} \mathbb{K}_\infty^+ & \to & \mathbb{R} \\ z & \mapsto & \frac{1}{2} \sum\limits_{\substack{z = z_1 + z_2 \\ x_1 + x_2 = x}} \left(\varphi_{x_1}(z_1)\psi'_{x_2}(z_2) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_2) \right) \\ & & + \frac{1}{2} \sum\limits_{\substack{z = z_1 - z_2 \\ x_1 + x_2 = x}} \left(\psi_{x_1}(z_1)\varphi'_{x_2}(z_2) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2) - \psi_{x_1}(z_2)\varphi'_{x_2}(z_1) + \varphi_{x_1}(z_2)\psi'_{x_2}(z_1) \right) \end{array} \right.$$

Remark 7.2.2. The terms in χ_x and χ'_x are all the terms that can appear in the usual trigonometric formulae.

Proposition 7.2.3. *Addition and multiplication are well defined.*

Proof. We use the notations of the above definition.

- For addition first. The sum of summable families is summable then $(\chi_x)_{x \in \mathbb{K}_\infty}$ and $(\chi'_x)_{x \in \mathbb{K}_\infty}$ are summable. Moreover $\operatorname{supp} \chi_x \cup \operatorname{supp} \chi'_x \subseteq \operatorname{supp} \varphi_x \cup \operatorname{supp} \varphi'_x \cup \operatorname{supp} \psi_x \cup \operatorname{supp} \psi'_x$. It is included in a reverse well-ordered subset, so it is reverse well-ordered. Moreover, it is also included in a free $\mathbb{Z}$-module of finite dimension by union.

- For multiplication, it is more complicated.

  (i) Since the families $(\varphi_{x_1}(z))_{z \in \mathbb{K}_\infty^+}$, $(\psi_{x_1}(z))_{z \in \mathbb{K}_\infty^+}$, $(\varphi'_{x_2}(z))_{z \in \mathbb{K}_\infty^+}$, $(\psi'_{x_2}(z))_{z \in \mathbb{K}_\infty^+}$ are summable, the families $(\varphi_{x_1}(z_1)\psi'_{x_2}(z_2))_{z_1, z_2 \in \mathbb{K}_\infty^+}$ and $(\psi_{x_1}(z_1)\varphi'_{x_2}(z_2))_{z_1, z_2 \in \mathbb{K}_\infty}$ are also summable.

  (ii) Since $\{ x \in \mathbb{K}_\infty \mid \operatorname{supp} \varphi_x \cup \sup \psi_x \neq \varnothing \}$ and $\{ x \in \mathbb{K}_\infty \mid \operatorname{supp} \varphi'_x \cup \sup \psi'_x \neq \varnothing \}$ are well-ordered, for all $x \in \mathbb{K}_\infty$, there are finitely many x_1 and x_2 such that

  $$x_1 + x_2 = x \quad \text{and} \quad \operatorname{supp} \varphi_{x_1} \cup \sup \psi_{x_1} \cup \operatorname{supp} \varphi'_{x_2} \cup \sup \psi'_{x_2} \neq \varnothing$$

  Therefore, to prove that χ_x is well defined, we can fix the values of x_1 and x_2. Thanks to point (i), the families

  $$\left(\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2) \right)_{z = z_1 + z_2} \quad \text{and} \quad \left(\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2) - \psi_{x_1}(z_1)\psi'_{x_2}(z_2) \right)_{z = z_1 - z_2}$$

  are summable when x_1 and x_2 are fixed. Therefore, the function χ_x is well defined for all $x \in \mathbb{K}_\infty$.

  (iii) Similarly, χ'_x is also well defined for all x.

  (iv) We can use point (i) again, we have that the families $\left(\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2) \right)_{z_1, z_2}$ and $\left(\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2) - \psi_{x_1}(z_1)\psi'_{x_2}(z_2) \right)_{z_1, z_2}$ are summable for x_1 and x_2 fixed. As in point (ii), there are finitely many x_1 and x_2 such that $x_1 + x_2 = x$ for x fixed. Therefore, the families

  $$\left(\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2) \right)_{z_1, z_2, x_1 + x_2 = x} \quad \text{and} \quad \left(\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2) - \psi_{x_1}(z_1)\psi'_{x_2}(z_2) \right)_{z_1, z_2, x_1 + x_2 = x}$$

  are themselves summable. Hence, $(\chi_x(z))_{z \in \mathbb{K}_\infty^+}$ is summable.

(v) Similarly, $(\chi'_x(z))_{z\in\mathbb{K}^+_\infty}$ is summable.

(vi) For all x, $\operatorname{supp}\chi_x \cup \operatorname{supp}\chi'_x$ is contained in a sum of free $\mathbb{Z}$-modules of finite dimension. Since $\mathbb{Z}$ is an Euclidean ring, this sum is also a free $\mathbb{Z}$-module of finite dimension.

(vii) Finally, assume x is such that $\operatorname{supp}\chi_x \cup \operatorname{supp}\chi'_x \neq \varnothing$. Then, there are x_1 and w_2 such that $x_1 + x_2 = x$, $\operatorname{supp}\varphi_{x_1} \cup \psi_{x_1} \neq \varnothing$, $\operatorname{supp}\varphi'_{x_2} \cup \operatorname{supp}\psi'_{x_2} \neq \varnothing$. Therefor $x \in X_1 + X_2$ for two reverse well-ordered sets X_1 and X_2. Therefore $\{x \in \mathbb{K}_\infty \mid \operatorname{supp}\chi_x \cup \operatorname{supp}\chi'_x \neq \varnothing\} \subseteq X_1 + X_2$. Proposition 2.4.3 concludes $\{x \in \mathbb{K}_\infty \mid \operatorname{supp}\chi_x \cup \operatorname{supp}\chi'_x \neq \varnothing\}$ is reverse well-ordered.

□

Proposition 7.2.4. $(\mathbb{O}_\mathbb{K}, +, \times)$ *is a ring such that*

$$0 = \sum_{x\in\mathbb{K}_\infty} \exp(x)(C(0) + S(0))$$

and

$$1 = \sum_{x\in\mathbb{K}_\infty} \exp(x)(C(\varphi_x) + S(0))$$

where

$$\varphi_x : \begin{cases} \mathbb{K}^+_\infty & \to & \mathbb{R} \\ z & \mapsto & \begin{cases} 1 & \textit{if } x = z = 0 \\ 0 & \textit{otherwise} \end{cases} \end{cases}$$

Proof. [1] The fact that $(\mathbb{O}_\mathbb{K}, +)$ is an Abelian group is inherited by the property of addition over the real numbers. Now for multiplication, consider

$$a = \sum_{x\in\mathbb{K}_\infty} \exp(x)(C(\varphi_x) + S(\psi_x)) \quad b = \sum_{x\in\mathbb{K}_\infty} \exp(x)(C(\varphi'_x) + S(\psi'_x)) \quad \text{and} \quad c = \sum_{x\in\mathbb{K}_\infty} \exp(x)(C(\varphi''_x) + S(\psi''_x))$$

- By symmetry of the role of a and b in the product ab, multiplication is commutative.

- 1 is neutral: Denote
$$a1 = \sum_{x\in\mathbb{K}_\infty} \exp(x)(C(\chi_x) + S(\chi'_x))$$
By definition of multiplication, we have for all $x \in \mathbb{K}_\infty$ and for all $z \in \mathbb{K}^+_\infty$,
$$\chi_x(z) = \varphi_x(z) \qquad \text{and} \qquad \chi'_x(z) = \psi_x(z)$$
Therefore 1 is indeed neutral.

- Let us denote $a(bc) = \sum_{x\in\mathbb{K}_\infty} \exp(x)(C(\chi_x) + S(\chi'_x))$. By definition of multiplication, for all $x \in \mathbb{K}_\infty$ and $z \in \mathbb{K}^+_\infty$, Let us now do a case disjunction and see which term contribute in what case. We fix a decomposition $x = x_1 + x_2 + x_3$ and look at the definition of multiplication. We look at the terms in the sum obtained by applying the definition of χ_x. We multiply by for so get rid of the $1/2$ factors.

Decomposition	Contribution in $4\chi_x(z)$
$z = z_1 + z_2 + z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) - \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = z_1 + \lvert z_2 - z_3\rvert$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$
$z = z_1 + z_2 - z_3$ $z_2 \geq z_3$	$-\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$
$z = \lvert z_1 - z_2 - z_3\rvert$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = \lvert z_1 - \lvert z_2 - z_3\rvert\rvert$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$
$z = \lvert z_1 - z_2 + z_3\rvert$ $z_2 \geq z_3$	$\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$

We split lines 2 and 4 on the cases $z_2 \geq z_3$ and $z_2 < z_3$.

[1] This is a painful proof which consists mainly in tedious computations which are detailed here.

Decomposition	Contribution in $4\chi_x(z)$
$z = z_1 + z_2 + z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) - \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = z_1 + z_2 - z_3$ $z_2 \geq z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$
$z = z_1 - z_2 + z_3$ $z_2 < z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$
$z = z_1 + z_2 - z_3$ $z_2 \geq z_3$	$-\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$
$z = \|z_1 - z_2 - z_3\|$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = \|z_1 - z_2 + z_3\|$ $z_2 \geq z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$
$z = \|z_1 + z_2 - z_3\|$ $z_2 < z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$
$z = \|z_1 - z_2 + z_3\|$ $z_2 \geq z_3$	$\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$

We now merge the lines 2 and 4 as well as lines 6 and 8 and immediately split them on case of equality.

Decomposition	Contribution in $4\chi_x(z)$
$z = z_1 + z_2 + z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) - \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = z_1 - z_2 + z_3$ $z_2 < z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$
$z = z_1$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_2) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_2)$
$z = z_1 + z_2 - z_3$ $z_2 > z_3$	$-\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$
$z = \|z_1 - z_2 - z_3\|$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = \|z_1 + z_2 - z_3\|$ $z_2 < z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$
$z = \|z_1\| = z_1$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_2) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_2)$
$z = \|z_1 - z_2 + z_3\|$ $z_2 > z_3$	$\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$

We merge lines 3 and 7 and switch the roles of z_2 and z_3 in lines 2 and 6.

Decomposition	Contribution in $4\chi_x(z)$
$z = z_1 + z_2 + z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) - \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = z_1 - z_3 + z_2$ $z_2 > z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_3)\psi''_{x_3}(z_2)$
$z = z_1$	$2\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_2) + 2\varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_2)$
$z = z_1 + z_2 - z_3$ $z_2 > z_3$	$-\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$
$z = \|z_1 - z_2 - z_3\|$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = \|z_1 + z_3 - z_2\|$ $z_2 > z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_3)\psi''_{x_3}(z_2)$
$z = \|z_1 - z_2 + z_3\|$ $z_2 > z_3$	$\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$

We now can merge lines 2 and 4 as well as lines 6 and 7.

Decomposition	Contribution in $4\chi_x(z)$
$z = z_1 + z_2 + z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) - \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = z_1$	$2\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_2) + 2\varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_2)$
$z = z_1 + z_2 - z_3$ $z_2 > z_3$	$-\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_3)\psi''_{x_3}(z_2)$
$z = \|z_1 - z_2 - z_3\|$ $= \|z_3 + z_2 - z_1\|$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = \|z_1 - z_2 + z_3\|$ $z_2 > z_3$	$\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_3)\psi''_{x_3}(z_2)$

We split the fourth line in the following cases:

- ➢ $z = z_1 - z_2 - z_3 > 0$
- ➢ $z = z_3$ and $z_2 = z_1$
- ➢ $z = z_3 + z_2 - z_1$ and $z_2 > z_1$
- ➢ $z = z_3 + z_2 - z_1$ and $z_2 < z_1$

We also split the fifth line into these ones:

- ➢ $z = z_2 - z_1 - z_3 > 0$
- ➢ $z = z_3 + z_1 - z_2$ and $z_2 > z_3$

Decomposition	Contribution in $4\chi_x(z)$
$z = z_1 + z_2 + z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) - \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = z_1$	$2\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_2) + 2\varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_2)$
$z = z_1 + z_2 - z_3$ $z_2 > z_3$	$-\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_3)\psi''_{x_3}(z_2)$
$z = z_1 - z_2 - z_3 > 0$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = z_3 + z_2 - z_1$ $z_2 > z_1$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = z_3 + z_2 - z_1$ $z_2 < z_1$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = z_3$	$\varphi_{x_1}(z_2)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_2)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_2)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) + \psi_{x_1}(z_2)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = z_2 - z_1 - z_3 > 0$	$\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_3)\psi''_{x_3}(z_2)$
$z = z_3 + z_1 - z_2$ $z_2 > z_3$	$\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_3)\psi''_{x_3}(z_2)$

We switch the roles of z_1 and z_3 in line 5 and merge it with line 3. We also rename z_3 in z_1 in line 7 and merge it with line 2. We change the roles of z_1 and z_2 in line 8 and merge it with line 4. Finally, in line 9, z_2 is renamed z_1, z_1 is renamed z_3 and z_3 is renamed z_2 so that we can merge it with line 6.

Decomposition	Contribution in $4\chi_x(z)$
$z = z_1 + z_2 + z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) - \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = z_1$	$2\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_2) + 2\varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_2)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_1) - \varphi_{x_1}(z_2)\psi'_{x_2}(z_2)\psi''_{x_3}(z_1)$ $+\psi_{x_1}(z_2)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_1) + \psi_{x_1}(z_2)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_1)$
$z = z_1 + z_2 - z_3$ $z_2 > z_3$	$-\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_3)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_1) - \varphi_{x_1}(z_3)\psi'_{x_2}(z_2)\psi''_{x_3}(z_1)$ $+\psi_{x_1}(z_3)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_1) + \psi_{x_1}(z_3)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_1)$
$z = z_1 - z_2 - z_3 > 0$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$ $+\psi_{x_1}(z_2)\psi'_{x_2}(z_1)\varphi''_{x_3}(z_3) - \psi_{x_1}(z_2)\varphi'_{x_2}(z_1)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_2)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_1) + \psi_{x_1}(z_2)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_1)$ $+\varphi_{x_1}(z_2)\varphi'_{x_2}(z_1)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_2)\psi'_{x_2}(z_1)\psi''_{x_3}(z_3)$ $+\varphi_{x_1}(z_2)\varphi'_{x_2}(z_3)\varphi''_{x_3}(z_1) + \varphi_{x_1}(z_2)\psi'_{x_2}(z_3)\psi''_{x_3}(z_1)$
$z = z_3 + z_2 - z_1$ $z_2 < z_1$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$ $+\psi_{x_1}(z_3)\psi'_{x_2}(z_1)\varphi''_{x_3}(z_2) - \psi_{x_1}(z_3)\varphi'_{x_2}(z_1)\psi''_{x_3}(z_2)$ $-\psi_{x_1}(z_3)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_1) + \psi_{x_1}(z_3)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_1)$ $+\varphi_{x_1}(z_3)\varphi'_{x_2}(z_1)\varphi''_{x_3}(z_2) + \varphi_{x_1}(z_3)\psi'_{x_2}(z_1)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_3)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_1) + \varphi_{x_1}(z_3)\psi'_{x_2}(z_2)\psi''_{x_3}(z_1)$

In line 4, z_2 and z_3 have identical role. We then can switch them independently on each term of the sum as we wish. We also exchange the roles of z_1 and z_3 in line 5.

Decomposition	Contribution in $4\chi_x(z)$
$z = z_1 + z_2 + z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) - \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = z_1$	$2\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_2) + 2\varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_2)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_1) - \varphi_{x_1}(z_2)\psi'_{x_2}(z_2)\psi''_{x_3}(z_1)$ $+\psi_{x_1}(z_2)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_1) + \psi_{x_1}(z_2)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_1)$
$z = z_1 + z_2 - z_3$ $z_2 > z_3$	$-\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_3)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_1) - \varphi_{x_1}(z_3)\psi'_{x_2}(z_2)\psi''_{x_3}(z_1)$ $+\psi_{x_1}(z_3)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_1) + \psi_{x_1}(z_3)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_1)$
$z = z_1 - z_2 - z_3 > 0$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$ $+\psi_{x_1}(z_2)\psi'_{x_2}(z_1)\varphi''_{x_3}(z_3) - \psi_{x_1}(z_2)\varphi'_{x_2}(z_1)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_3)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_1) + \psi_{x_1}(z_3)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_1)$ $+\varphi_{x_1}(z_2)\varphi'_{x_2}(z_1)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_2)\psi'_{x_2}(z_1)\psi''_{x_3}(z_3)$ $+\varphi_{x_1}(z_3)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_1) + \varphi_{x_1}(z_3)\psi'_{x_2}(z_2)\psi''_{x_3}(z_1)$
$z = z_1 + z_2 - z_3$ $z_2 < z_3$	$\varphi_{x_1}(z_3)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_1) - \varphi_{x_1}(z_3)\psi'_{x_2}(z_2)\psi''_{x_3}(z_1)$ $+\psi_{x_1}(z_3)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_1) + \psi_{x_1}(z_3)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_1)$ $+\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $-\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_1)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_3)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$

Notice that lines 2 and 5 contribute the very same way. We then can merge the two cases into a single one $z = z_1 + z_2 - z_3$ substracting the case $z_2 = z_3$ and $z = z_1$. We merge this new case with the line 2, what turns out to cancel this line.

Decomposition	Contribution in $4\chi_x(z)$
$z = z_1 + z_2 + z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) - \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = z_1$	0
$z = z_1 + z_2 - z_3$	$-\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_3)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_1) - \varphi_{x_1}(z_3)\psi'_{x_2}(z_2)\psi''_{x_3}(z_1)$ $+\psi_{x_1}(z_3)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_1) + \psi_{x_1}(z_3)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_1)$
$z = z_1 - z_2 - z_3 > 0$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$ $+\psi_{x_1}(z_2)\psi'_{x_2}(z_1)\varphi''_{x_3}(z_3) - \psi_{x_1}(z_2)\varphi'_{x_2}(z_1)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_3)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_1) + \psi_{x_1}(z_3)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_1)$ $+\varphi_{x_1}(z_2)\varphi'_{x_2}(z_1)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_2)\psi'_{x_2}(z_1)\psi''_{x_3}(z_3)$ $+\varphi_{x_1}(z_3)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_1) + \varphi_{x_1}(z_3)\psi'_{x_2}(z_2)\psi''_{x_3}(z_1)$

We remove line 1 which does not contribute and simplify in line 3.

Decomposition	Contribution in $4\chi_x(z)$
$z = z_1 + z_2 + z_3$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) - \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$
$z = z_1 + z_2 - z_3$	$+\psi_{x_1}(z_1)\psi'_{x_2}(z_3)\varphi''_{x_3}(z_2) - \psi_{x_1}(z_1)\varphi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\varphi_{x_1}(z_1)\varphi'_{x_2}(z_3)\varphi''_{x_3}(z_2) + \varphi_{x_1}(z_1)\psi'_{x_2}(z_3)\psi''_{x_3}(z_2)$ $+\varphi_{x_1}(z_3)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_1) - \varphi_{x_1}(z_3)\psi'_{x_2}(z_2)\psi''_{x_3}(z_1)$ $+\psi_{x_1}(z_3)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_1) + \psi_{x_1}(z_3)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_1)$
$z = z_1 - z_2 - z_3 > 0$	$\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_3) - \varphi_{x_1}(z_1)\psi'_{x_2}(z_2)\psi''_{x_3}(z_3)$ $+\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_3) + \psi_{x_1}(z_1)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_3)$ $+\psi_{x_1}(z_2)\psi'_{x_2}(z_1)\varphi''_{x_3}(z_3) - \psi_{x_1}(z_2)\varphi'_{x_2}(z_1)\psi''_{x_3}(z_3)$ $-\psi_{x_1}(z_3)\psi'_{x_2}(z_2)\varphi''_{x_3}(z_1) + \psi_{x_1}(z_3)\varphi'_{x_2}(z_2)\psi''_{x_3}(z_1)$ $+\varphi_{x_1}(z_2)\varphi'_{x_2}(z_1)\varphi''_{x_3}(z_3) + \varphi_{x_1}(z_2)\psi'_{x_2}(z_1)\psi''_{x_3}(z_3)$ $+\varphi_{x_1}(z_3)\varphi'_{x_2}(z_2)\varphi''_{x_3}(z_1) + \varphi_{x_1}(z_3)\psi'_{x_2}(z_2)\psi''_{x_3}(z_1)$

In line line 1, z_1 and z_3 have identical role. Therefore φ''_{x_3} and φ_{x_1} can be switched with no harm. In line 2, z_1 and z_2 have the same role. When either φ_{x_1} or φ''_{x_3} is applied to z_3, the symmetric case is also in the contribution. Hence, we can again switch φ_{x_1} and φ''_{x_3}. The same phenomenon occurs in line 3. Therefore, φ''_{x_3} and φ_{x_1} have the same role in the definition of χ_x. The same work can be done for χ'_x. Then we conclude that $a(bc) = c(ba)$. Using commutativity, we conclude to the associativity of the multiplication law.

- Finally, distributivity is directly inherited from distributivity over real numbers.

$(\mathbb{O}_{\mathbb{K}}, +, \times)$ is a commutative ring. □

Chapter 8

Conclusion

In this thesis we have worked around the stability of fields of surreal numbers under operations such as the exponential, the logarithm, the derivation and the anti-derivation. We have done this to get fields on which we will be able to work when we see surreal numbers as dynamical systems. More generally, our aim was try to solve any polynomial ordinary differential equation in the context of oscillating numbers. Therefore, we need to get stability by all the operations mentioned above. We have done the work for surreal numbers, but we think that there is a similar result for oscillating numbers.
Note also that polynomial differential equations have been studied for complex transseries (see van der Hoven's work [49, 50]). However, we insist on the fact that some computable functions are too big to be characterized with transseries and may need a vector of polynomial ordinary differential equations. This means that the context is different and we cannot directly apply van der Hoeven's work.
This chapter is the final one of this thesis. We make a quick summary and give some ideas for future works.

- Section 8.1 recalls the major results of this thesis.
- Section 8.2 is dedicated to work in progress and perspective about surreal numbers an oscillating numbers.

8.1 Summary of contributions

We recall our main results. We follows the same plot as the thesis itself.

8.1.1 Stability of special surreal subfields

In this thesis, we have defined first surreal subfields that are stable under exponential and logarithm as these functions are essentials to the polynomial differential equations we want to solve.

Definition 5.1.5. Let Γ be an Abelian subgroup of **No** and λ be an ε-number. Let α such that $\lambda = \varepsilon_\alpha$. We have

$$\lambda = \sup E_\lambda$$

where
$$E_\lambda = \begin{cases} \{\varepsilon_\beta \uparrow\uparrow n \mid n \in \mathbb{N}\} & \beta + 1 = \alpha \\ \{1\} \cup \{\varepsilon_\beta \mid \beta < \alpha\} & \alpha \in \mathbf{Lim} \end{cases}$$

and $\uparrow\uparrow$ is the Knuth's arrow notation. Namely,

$$x \uparrow\uparrow 0 = 1 \qquad \text{and} \qquad \forall n \in \mathbb{N} \quad x \uparrow\uparrow (n+1) = x^{x \uparrow\uparrow n}$$

In other words,
$$x \uparrow\uparrow n = \left. x^{x^{\cdot^{\cdot^{\cdot^{x}}}}} \right\} n \text{ occurences of } x$$

We may write $E_\lambda = (e_\beta)_{\beta < \gamma_\lambda}$ where $\gamma_\lambda = \begin{cases} \omega & \beta + 1 = \alpha \\ \alpha & \alpha \in \mathbf{Lim} \end{cases}$

We denote by $\Gamma^{\uparrow\lambda}$ the family of group $(\Gamma_\beta)_{\beta < \gamma_\lambda}$ defined as follows :

- $\Gamma_0 = \Gamma$
- $\Gamma_{\beta+1}$ is the group generated by Γ_β, $\mathbb{R}_{e_\beta}^{g\left((\Gamma_\beta)_+^*\right)}$ and $\left\{ h(a_i) \;\middle|\; \sum\limits_{i<\nu} r_i \omega^{a_i} \in \Gamma_\alpha \right\}$
- For limit ordinal numbers β, $\Gamma_\beta = \bigcup\limits_{\gamma < \beta} \Gamma_\gamma$.

Theorem 5.1.6. *Let Γ be an Abelian subgroup of* **No** *and λ be an ε-number, then $\mathbb{R}_\lambda^{\Gamma^{\uparrow\lambda}}$ (see Definition 3.3.10) is stable under exponential and logarithmic functions.*

This theorem is one of our main contribution but is actually a particular case of the following proposition:

Proposition 5.1.7. *Let λ be an ε-number and $(\Gamma_i)_{i\in I}$ be a family of Abelian subgroups of* **No**. *Then $\mathbb{R}_\lambda^{(\Gamma_i)_{i\in I}}$ is stable under* exp *and* ln *if and only if*

$$\bigcup_{i\in I}\Gamma_i = \bigcup_{i\in I}\mathbb{R}_\lambda^{g\left((\Gamma_i)_+^*\right)}$$

After proving this result, we made the link between $\mathbf{No}_\lambda$, which is stable under exp and ln and fields of the form $\mathbb{R}_\lambda^{\Gamma^\uparrow}$. More precisely, we decomposed $\mathbf{No}_\lambda$ into a hierarchy of subfields which is strict.

Theorem 5.1.10. $\mathbf{No}_\lambda = \bigcup_\mu \mathbb{R}_\lambda^{{\mathbf{No}_\mu}^{\uparrow\lambda}}$*, where μ ranges over the additive ordinals less than λ (equivalently, μ ranges over the multiplicative ordinals less λ),*

Theorem 5.1.11. *For all ε-number λ, the hierarchy in previous theorem is strict:*

$$\mathbb{R}_\lambda^{{\mathbf{No}_\mu}^{\uparrow\lambda}} \subsetneq \mathbb{R}_\lambda^{{\mathbf{No}_{\mu'}}^{\uparrow\lambda}}$$

for all multiplicative ordinals μ and μ' such that $\omega < \mu < \mu' < \lambda$.

Subfields stable under exp and ln being defined, we characterized some subfields stable under exp, ln, derivation and anti-derivation.

Theorem 5.3.1. *Let α be a limit ordinal and $(\Gamma_\beta)_{\beta<\alpha}$ be a sequence of Abelian subgroups of* **No** *such that*

- $\forall\beta<\alpha \quad \forall\gamma<\beta \qquad \Gamma_\gamma\subseteq\Gamma_\beta$
- $\forall\beta<\alpha \qquad \omega^{(\Gamma_\beta)_+^*} \succ^K \kappa_{-\varepsilon_\beta}$
- $\forall\beta<\alpha \quad \forall\gamma<\varepsilon_\beta \qquad \kappa_{-\gamma}\in\omega^{\Gamma_\beta}$
- $\forall\beta<\alpha \quad \exists\eta_\beta<\varepsilon_\beta \quad \forall x\in\omega^{\Gamma_\beta} \qquad \mathrm{NR}(x)<\eta_\beta$

Then $\bigcup_{\beta<\alpha}\mathbb{R}_{\varepsilon_\beta}^{\Gamma_\beta^{\uparrow\varepsilon_\beta}}$ *is stable under* exp, ln, ∂ *and anti-derivation.*

Using this theorem, we concluded our contribution about surreal subfield by showing Example 5.3.3 that the field $\bigcup_{n\in\mathbb{N}}\mathbb{R}_{\varepsilon_n}^{\mathbf{No}_{\varepsilon_n}^{\uparrow\varepsilon_n}}$ is stable under exp, ln, derivation and anti-derivation. Recall that this was not a direct application of the theorem: we had to rewrite it properly to be able to apply the theorem. The idea of this example enabled Matusinki and the author of this thesis to state that the last condition of Theorem 5.3.1 is in fact not necessary.

Corollary 5.3.2 (Guilmant-Matusinski)**.** *Let α be a limit ordinal and $(\Gamma_\beta)_{\beta<\alpha}$ be a sequence of Abelian subgroups of* **No** *such that*

- $\forall\beta<\alpha \quad \forall\gamma<\beta \qquad \Gamma_\gamma\subseteq\Gamma_\beta$
- $\forall\beta<\alpha \qquad \omega^{(\Gamma_\beta)_+^*} \succ^K \kappa_{-\varepsilon_\beta}$
- $\forall\beta<\alpha \quad \forall\gamma<\varepsilon_\beta \qquad \kappa_{-\gamma}\in\omega^{\Gamma_\beta}$

Then $\bigcup_{\beta<\alpha}\mathbb{R}_{\varepsilon_\beta}^{\Gamma_\beta^{\uparrow\varepsilon_\beta}}$ *is stable under* exp, ln, ∂ *and anti-derivation.*

8.1.2 Topological aspects

Chapter 6 is dedicated to topological aspects of surreal numbers. The goal was to contribute to some notions that we think will be helpful to apply a kind of Intermediate Value Theorem with the derivation, ∂. To do so, we investigated the problems that causes the usual definition of continuity and fixed it with the notion of gap-continuity. Since continuity is related to compactness, we developed a counterpart in the context of surreal number to be handle with this notion of gap-continuity.

Definition 6.2.5 ((λ,Γ)-gap-compact set)**.** If $\mathcal{X}$ is a set of open intervals of $\widetilde{\mathbb{R}_\lambda^\Gamma}$, let $\mathcal{B}(\mathcal{X})$ the set of the bounds of theses intervals. Now, a subset $X\subseteq\widetilde{\mathbb{R}_\lambda^\Gamma}$ is said (λ,Γ)-gap-compact if any covering $\mathcal{X}$ of X by open intervals such that for all non-trivial gap $\langle L \dashv\vdash R\rangle$ such that $L\cup R=\mathcal{B}(\mathcal{X})$, there is $I\in\mathcal{X}$ such that $\inf I\in L$ and $\sup I\in R$ admits a finite sub-covering. Written with a mathematical formula:

$$\forall\,\langle L\dashv\vdash R\rangle\in\mathcal{G}_\perp\widetilde{\mathbb{R}_\lambda^\Gamma}\quad L\cup R=\mathcal{B}(\mathcal{X})\quad(\exists I\in\mathcal{X}\quad\inf I\in L\wedge\sup I\in R)\Rightarrow\left(\exists\mathcal{X}'\subseteq\mathcal{X}\quad|\mathcal{X}'|<\infty\wedge X\subseteq\bigcup_{I\in\mathcal{X}'}I\right)$$

Definition 6.2.7 (Gap-connected set). $X \subseteq \widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$ is said to be **gap-connected** if for any non-trivial gap $\langle L \dashv\vdash R\rangle$ such that $L \subseteq X$ or $R \subseteq X$, there are $L', R' \subseteq X$ such that $\langle L \dashv\vdash R\rangle = \langle L' \dashv\vdash R'\rangle$.

With these notion, we can give a characterization that is very closed to the real case.

Proposition 6.2.10. *$X \subseteq \widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$ is (λ,Γ)-gap-compact if and only if X is bounded closed and gap-connected.*

We also extended the definition of gap-compactness considering any open set instead of solely open intervals.

Proposition 6.2.16. *X is (λ,Γ)-gap-compact if an only if from any covering $\mathcal{X}$ of X by open set such that for any non-trivial gap $\langle L \dashv\vdash R\rangle$ such that $L \cup R = \mathcal{B}_{set}(\mathcal{X})$, there is $U \in \mathcal{X}$ that is a neighborhood of $\langle L \dashv\vdash R\rangle$, we can extract a finite subcovering.*

After that, we could take a closer look to the notion of gap-continuity.

Definition 6.3.2. A function $f : \widetilde{\mathbb{R}^{\Gamma}_{\lambda}} \to \widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$ is said to be (λ,Γ)-**gap-continuous** if it satisfies:

GC1. f is continuous (*i.e* satisfies Definition 6.0.1).

GC2. For any non-trivial gap $G = \langle L \dashv\vdash R\rangle \in \mathcal{G}_{\perp}\widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$, there is some $y \in \widetilde{\mathbb{R}^{\Gamma}_{\lambda}} \cup \mathcal{G}_{\perp}\widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$ such that for any neighborhood J of y, there is some neighborhood I of G such that

$$x \in I \implies f(x) \in J$$

Moreover, if $y \in \widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$ then we must have $y \in f(I)$.

GC3. For any non-trivial gap $G = \langle L \dashv\vdash R\rangle \in \mathcal{G}_{\perp}\widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$, if y given by **GC2.** is a gap, for any neighborhood I of G

$$f(I) \cap \left\{ z \in \widetilde{\mathbb{R}^{\Gamma}_{\lambda}} \;\middle|\; z > y \right\} \neq \varnothing \qquad \text{and} \qquad f(I) \cap \left\{ z \in \widetilde{\mathbb{R}^{\Gamma}_{\lambda}} \;\middle|\; z < y \right\} \neq \varnothing$$

f is (λ,Γ)-**weakly-gap-continuous** if it satisfies **GC1.** and **GC2.**.

With this definition, we could prove the following theorems:

Theorem 6.3.8 (Intermediate value theorem). *Let $f : \widetilde{\mathbb{R}^{\Gamma}_{\lambda}} \to \widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$ be (λ,Γ)-weakly-gap-continuous and $a < b$ such that assume $f(a) \leq f(b)$. Then, for all $y \in [\min(f(a), f(b))\,;\max(f(a), f(b))]$ there is $a \leq c \leq b$ such that $f(c) = y$.*

Theorem 6.3.9. *Let $f : \widetilde{\mathbb{R}^{\Gamma}_{\lambda}} \to \widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$ be a (λ,Γ)-gap-continuous function. Let $X \subseteq \widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$ be a (λ,Γ)-gap-compact set, then $f(X)$ is also (λ,Γ)-gap-compact .*

Theorem 6.3.10 (Extreme values theorem). *Let $f : \widetilde{\mathbb{R}^{\Gamma}_{\lambda}} \to \widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$ be a (λ,Γ)-gap-continuous function. Let $X \subseteq \widetilde{\mathbb{R}^{\Gamma}_{\lambda}}$ be (λ,Γ)-gap-compact . Then f reaches its extrema on X.*

8.1.3 Oscillating numbers

In Chapter 7, we developed the notion of oscillating numbers. The chapter is entirely dedicated to this new structure and prove that this is a ring. Taking its field of fractions, we get a field.

Definition 7.1.1. Let $\mathbb{K}$ a field of surreal numbers that is stable under $\exp$ and $\ln$. Let $\mathbb{K}_{\infty}$ the set (or class) of the purely infinite numbers in $\mathbb{K}$. Notice that $\mathbb{K}$ must contain $\mathbb{Q}$ and therefore, $\mathbb{K}_{\infty}$ is a divisible group. Let $\mathbb{K}_{\infty}^{+}$ the set of non-negative elements in $\mathbb{K}_{\infty}$. The set (or class) of **oscillating numbers** over $\mathbb{K}$ is the set (or class):

$$\mathbb{O}_{\mathbb{K}} = \left\{ \sum_{x\in\mathbb{K}_{\infty}} \exp(x)(C(\varphi_x) + S(\psi_x)) \;\middle|\; \begin{array}{c} \varphi_x : \mathbb{K}_{\infty}^{+} \to \mathbb{R} \quad \text{and} \quad \psi_x : \mathbb{K}_{\infty}^{+} \to \mathbb{R}, \psi_x(0) = 0 \\ (\varphi_x(z))_{z\in\mathbb{K}_{\infty}^{+}} \text{ and } (\psi_x(z))_{z\in\mathbb{K}_{\infty}^{+}} \text{ are summable} \\ \{x \in \mathbb{K}_{\infty} \mid \operatorname{supp}\varphi_x \cup \operatorname{supp}\psi_x \neq \varnothing\} \text{ is reverse well-ordered} \\ \operatorname{supp}\varphi_x \cup \operatorname{supp}\psi_x \text{ is contained in a free } \mathbb{Z}\text{-module of finite dimension} \end{array} \right\}$$

The operation over osculating numbers is defined as followed:

Definition 7.2.1. Consider two oscillating numbers follows:

$$a = \sum_{x\in\mathbb{K}_{\infty}} \exp(x)(C(\varphi_x) + S(\psi_x)) \qquad \text{and} \qquad b = \sum_{x\in\mathbb{K}_{\infty}} \exp(x)(C(\varphi'_x) + S(\psi'_x))$$

We define the addition and multiplication operations as follows:

$$a+b=\sum_{x\in\mathbb{K}_\infty}\exp(x)(C(\chi_x)+S(\chi'_x))$$

where $\chi_x:\begin{cases}\mathbb{K}_\infty^+ & \to & \mathbb{R}\\ z & \mapsto & \varphi_x(z)+\varphi'_x(z)\end{cases}$ and $\chi'_x:\begin{cases}\mathbb{K}_\infty^+ & \to & \mathbb{R}\\ z & \mapsto & \psi_x(z)+\psi'_x(z)\end{cases}$

$$ab=\sum_{x\in\mathbb{K}_\infty}\exp(x)(C(\chi_x)+S(\chi'_x))$$

where

$$\chi_x:\begin{cases}\mathbb{K}_\infty^+ \to \mathbb{R}\\ z \mapsto \dfrac{1}{2}\displaystyle\sum_{\substack{z=z_1+z_2\\x_1+x_2=x}}\big(\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)-\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\big)\\ \qquad +\dfrac{1}{2}\displaystyle\sum_{\substack{z=|z_1-z_2|\\x_1+x_2=x}}\big(\varphi_{x_1}(z_1)\varphi'_{x_2}(z_2)+\psi_{x_1}(z_1)\psi'_{x_2}(z_2)\big)\end{cases}$$

and

$$\chi'_x:\begin{cases}\mathbb{K}_\infty^+ \to \mathbb{R}\\ z \mapsto \dfrac{1}{2}\displaystyle\sum_{\substack{z=z_1+z_2\\x_1+x_2=x}}\big(\varphi_{x_1}(z_1)\psi'_{x_2}(z_2)+\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)\big)\\ \qquad +\dfrac{1}{2}\displaystyle\sum_{\substack{z=z_1-z_2\\x_1+x_2=x}}\big(\psi_{x_1}(z_1)\varphi'_{x_2}(z_2)-\varphi_{x_1}(z_1)\psi'_{x_2}(z_2)-\psi_{x_1}(z_2)\varphi'_{x_2}(z_1)+\varphi_{x_1}(z_2)\psi'_{x_2}(z_1)\big)\end{cases}$$

Proposition 7.2.4. $(\mathbb{O}_\mathbb{K},+,\times)$ *is a ring such that*

$$0=\sum_{x\in\mathbb{K}_\infty}\exp(x)(C(0)+S(0))$$

and

$$1=\sum_{x\in\mathbb{K}_\infty}\exp(x)(C(\varphi_x)+S(0))$$

where

$$\varphi_x:\begin{cases}\mathbb{K}_\infty^+ & \to & \mathbb{R}\\ z & \mapsto & \begin{cases}1 & \textit{if } x=z=0\\ 0 & \textit{otherwise}\end{cases}\end{cases}$$

We conjecture that the ring of oscillating and its field of fraction can embedded the solution to any polynomial differential equation. This would enable us to see them as algorithm, an more precisely, as asymptotic behavior of GPACs.

8.2 Perspectives

We finish this monograph presenting some perspectives for future works. These perspectives strongly rely on oscillating numbers.

8.2.1 Solving ordinary polynomial differential equations

The function ∂ is not weakly gap-continuous. However, the gaps where Definition 6.2.5 fails are only the gaps $x\pm\dfrac{1}{+\infty}$. Moreover, we strongly believe that it satisfies the conclusion of the Intermediate Value Theorem. Therefore, we will try, in the future, to weaken again de definition of gap-continuity to take into account the specificity of ∂ while these changes still enable the Intermediate Value Theorem to be proved. This would lead to the possibility to get lot of solutions to differential equations of the form $\partial x=p(x)$ where p is some polynomial with surreal coefficients.

If we are able to so, it will be possible to go back to our original motivation, related to GPAC (General Purpose Analog Computer). Indeed, equation of the form $\partial x=p(x)$ may represent the behavior of a GPAC. Actually, for this special kind of equation, the GPAC will be very simple since there is only one dimension.

To try to solve the multi-dimension case, we must generalize the theorems we saw to oscillating numbers. This would require an exponential over oscillating numbers.

8.2.2 Exponential over oscillating number

To be able so solve equations of the form $\partial x=(\partial y)x$, where y a fixed osculating number and x the variable that may take values in the oscillating numbers, we need an exponential over oscillating numbers. This equation is very legitimate, therefore we have no choice but to define such an exponentiation.

In this section we try to explain what is our hope to define the exponentiation of oscillating numbers. This idea strongly relies on the modified Bessel function of the first kind.

Modified Bessel functions of the first kind

The reason why we need the modified Bessel function is because we want to generalize the following known identity:

$$\forall a, x \in \mathbb{R} \qquad \exp(a\cos x) = I_0(a) + 2\sum_{n=1}^{+\infty} I_n(a)\cos(nx)$$

where I_n is the modified Bessel function of the first kind defined by

$$\begin{aligned}\forall a \in \mathbb{R} \qquad I_n(a) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(a\cos t)\cos(nt)\,\mathrm{d}t \\ &= \frac{a^n}{2^{n-1}\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}\int_0^1 \left(1-t^2\right)^{n-\frac{1}{2}}\cosh(at)\,\mathrm{d}t \\ &= \left(\frac{a}{2}\right)^n \sum_{m=0}^{+\infty} \frac{a^{2m}}{4^m m!(m+n)!}\end{aligned}$$

In other words $I_n(a)$ is the half of the nth Fourier coefficient of $\exp(a\cos(x))$, except for $n = 0$, for which it is the Fourier coefficient itself. Using the identity $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ we can also deduce that

$$\forall a, x \in \mathbb{R} \qquad \exp(a\sin x) = I_0(a) + 2\sum_{n=0}^{+\infty}(-1)^n I_{2n+1}(a)\sin\left((2n+1)x\right) + 2\sum_{n=1}^{+\infty}(-1)^n I_{2n}(a)\cos(2nx)$$

For an introduction about Bessel functions, see for instance Waston's book about Bessel functions [52] or Appendix A for a very brief presentation of the basic properties.

Asymptotic expansion of I_n

To have a good generalization, we want to have access to the value of I_n for infinitely large numbers. To do so, we would like to try to use the asymptotic development of I_n.

Definition 8.2.1. Let f be a function over real numbers. Assume that there is a sequence $(a_k)_{k\in\mathbb{N}}$ and a function g such that

$$\forall K \in \mathbb{N} \qquad f(x) = g(x)\left(\sum_{k=0}^{K-1}\frac{a_k}{x^k} + O_{+\infty}\left(\frac{1}{x^K}\right)\right)$$

Then we write
$$f(x) \simeq g(x)\sum_{k=0}^{+\infty}\frac{a_k}{x^k}$$
and call this writing it an **asymptotic expansion** of the function f.

Remark 8.2.2. Asymptotic expansions may involve a power series $\sum a_k z^k$ that has a radius of convergence equal to 0.

That is why we use a symbol $\simeq$ since there may be no equality at all and the right member may exist only from a formal point of view.

Asymptotic expansions have very nice properties. To get an introduction about asymptotic expansions we suggest for instance Olver's book [37].
We do know an asymptotic expansion of $I_n(a)$ for $a \to \infty$.

Proposition 8.2.3 (See for instance [52, Watson, paragraph 7.23])**.** *We have*

$$I_n(a) \simeq \frac{\exp(a)}{\sqrt{2\pi a}}\sum_{k=0}^{+\infty}(-1)^k\frac{h_{0,n,k}}{a^k}$$

where
$$h_{0,n,k} = \frac{\prod_{i=1}^{k}\left(4n^2-(2i-1)^2\right)}{k!8^k}$$

Note that, as anticipated in Remark 8.2.2, the power series in the asymptotic expansion of $I_n(a)$ has radius of convergence equal to 0. Thus, there is equality for no real number a. However, in the context of infinite surreal numbers, this expression makes sens and we may make use of it to extend the definition of I_n.

Multi-index, multi-variable modified Bessel function In the modified Bessel function, there is only one variable inside the cosine function. However, in our definition of an oscillating numbers, we allow an arbitrarily large finite number of generators which can be seen as individual variables. We then introduce a generalization of the modified Bessel function. This paragraph is inspired by Stein's book about harmonic analysis [45, chapter 8]. We try to do the same work except that we do not include the complex number **i** in the exponential of the following integral and that we include the idea of the previous paragraph. Consider

$$I_k(a) = \int_{[\,0\,;\,1\,]^N} \exp\left(\sum_{n\in\mathbb{Z}^N} a_n \cos(2\pi n\cdot x)\right) \cos(2\pi k\cdot x)\,\mathrm{d}x$$

for $a \in \mathbb{R}^{\mathbb{Z}^N}$ and $k \in \mathbb{Z}^N$ for some finite set I. Here the operator $\cdot$ is the standard Euclidean scalar product. Notice that due to periodicity, we can take any segment of length 1 instead of $[\,0\,;1\,]$. Therefore,

$$I_k(a) = \int_{[\,-\frac{1}{4}\,;\,\frac{3}{4}\,]^N} \exp\left(\sum_{n\in\mathbb{Z}^N} a_n \cos(2\pi n\cdot x)\right) \cos(2\pi k\cdot x)\,\mathrm{d}x$$

Let
$$\mathcal{J} = \prod_{i\in I}\left\{\left[-\frac{1}{4}\,;\frac{1}{4}\right],\left[\frac{1}{4}\,;\frac{3}{4}\right]\right\}$$

so that
$$I_k(a) = \sum_{J\in\mathcal{J}}\int_J \exp\left(\sum_{n\in\mathbb{Z}^N} a_n \cos(2\pi n\cdot x)\right) \cos(2\pi k\cdot x)\,\mathrm{d}x$$

This partition is an taken from Stein's book [45, chapter 8]. Indeed, we think that this partition leads to several terms in the asymptotic development of this the function I_n.
Let $J\in\mathcal{J}$. We let $x_J = (x_{J,i})_{i\in[\![\,1\,;\,N\,]\!]}$ be the unique element such that

$$\{x_J\} = J\cap\left\{0,\frac{1}{2}\right\}^N$$

and $n_J = (n_{J,i})_{i\in[\![\,1\,;\,N\,]\!]}$ such that $n_{J,i} = \mathbb{1}_{x_{J,i}\neq 0}$ for $i\in[\![\,1\,;\,N\,]\!]$. Note that

$$\sin(2\pi n\cdot x_J) = 0 \qquad \text{and} \qquad \cos(2\pi n\cdot x_J) = (-1)^{n\cdot n_J}\in\{\pm 1\}^N$$

Denote
$$I_{k,J}(a) = \int_J \exp\left(\sum_{n\in\mathbb{Z}^N} a_{i,n}\cos(2\pi n x_i)\right)\cos(2\pi k\cdot x)\,\mathrm{d}x$$

Consider the change of variables $y = x - \dfrac{n_J}{2} = x - x_J$. Then,

$$\begin{aligned} I_{k,J}(a) &= \int_{[\,-\frac{1}{4}\,;\,\frac{1}{4}\,]^N} \exp\left(\sum_{n\in\mathbb{Z}^N} a_n\cos\left(2\pi n\cdot\left(y+\frac{n_J}{2}\right)\right)\right)\cos\left(2\pi k\cdot\left(y+\frac{n_J}{2}\right)\right)\,\mathrm{d}y \\ &= (-1)^{k\cdot n_J}\int_{[\,-\frac{1}{4}\,;\,\frac{1}{4}\,]^N} \exp\left(\sum_{n\in\mathbb{Z}^N} a_n(-1)^{n\cdot n_J}\cos\left(2\pi n\cdot y\right)\right)\cos(2\pi k\cdot y)\,\mathrm{d}y \end{aligned}$$

Definition of the exponentiation

Let $y\in\mathbb{O}_{\mathbb{K}^{(n)}}$. Write $y = \sum_{x\in\mathbb{K}_\infty^{(n)}}\exp(x)(C(\varphi_x)+S(\psi_x))$. Let also

$$y_\infty = \sum_{x\in\mathbb{K}_\infty^{(n)},x>0}\exp(x)(C(\varphi_x)+S(\psi_x)) \qquad \text{and} \qquad y_a = \sum_{x\in\mathbb{K}_\infty^{(n)},x\leq 0}\exp(x)(C(\varphi_x)+S(\psi_x))$$

We can define
$$\exp(y_a) = \sum_{k=0}^{+\infty}\frac{y_a^k}{k!} = \sum_{x\in\mathbb{K}_\infty^{(n)},x\leq 0}\exp(x)\left(\sum_{k=0}^{+\infty}\frac{1}{k!}\sum_{x_1+\cdots+x_k=x}\prod_{i=1}^{k}\left(C(\varphi_{x_i})+S(\psi_{x_i})\right)\right)$$

This makes sense because the right-most sum symbol in the above expression is actually a finite sum. The only that remain is so define $\exp(y_\infty)$ and then we will be able to set

$$\exp(y) = \exp(y_\infty)\exp(y_a)$$

We of course expect the exponential to satisfy its expected properties:

$$\forall y,z\in\mathbb{O}_{\mathbb{K}^{(n)}} \quad \exp(y)\exp(z) = \exp(y+z) \qquad \text{and} \qquad \exp(0) = 1$$

To solve this problem, we think that we may use the asymptotic development of the functions of form $t\mapsto I_n(ta)$ for $n\in\mathbb{Z}^N$ and $a\in\mathbb{R}^{\mathbb{Z}^N}$. With such an asymptotic development, we may be able to replace, up to some more work, the variable t by $\exp(x)$ for $x\in(\mathbb{K}_\infty)_+^*$. The vector a will be given by $C(\varphi_x)$.

8.2.3 Last note

Solving polynomial differential equations related to GPACs in an asymptotic view was the initial motivation of this thesis. We could not give a definitive answer to this question but we could identify some good surreal fields to work with. These fields, in turn, will help us to build good oscillating numbers fields on which we believe it will be possible to characterize the asymptotic behavior of GPACs.

Appendix A

Bessel function formulary

For $n \in \mathbb{R}$, we consider the function $\quad I_n(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \exp(x\cos t)\cos(nt)\,\mathrm{d}t$

This is the modified Bessel function of the first kind. In this appendix, we take a small look at some of it basic properties.

Lemma A.1. *For all $x \in \mathbb{R}$ and $n \in \mathbb{R}$, $I_n(x) = I_{-n}(x)$.*

Proof. Immediate from the parity of cos. □

Lemma A.2. *For all $x \in \mathbb{R}$ and $n \in \mathbb{R}$,* $\quad I_n'(x) = \dfrac{1}{2}\left(I_{n+1}(x) + I_{n-1}(x)\right)$

Proof sketch. We must check that we can indeed derive under the integral symbol. We then get

$$I_n'(x) = \frac{1}{\pi}\int_0^\pi \exp(x\cos t)\cos(nt)\cos t\,\mathrm{d}t$$

We then conclude using the trigonometric formula

$$2\cos(nt)\cot t = \cos((n+1)t) + \cos((n-1)t)$$

□

Lemma A.3. *For all $x \in \mathbb{R}$ and $n \in \mathbb{R}$,* $\quad 2nI_n(x) = x\left(I_{n-1}(x) - I_{n+1}(x)\right)$

Proof. We have

$$\begin{aligned}
x\left(I_{n-1}(x) - I_{n+1}(x)\right) &= \frac{1}{\pi}\int_0^\pi x\exp(x\cos t)\left(\cos((n-1)t) - \cos((n+1)t)\right)\,\mathrm{d}t \\
&= \frac{2}{\pi}\int_0^\pi x\exp(x\cos t)\sin(nt)\sin t\,\mathrm{d}t && \text{(formula for } 2\sin(a)\sin(b)\text{)} \\
&= \frac{2}{\pi}\left(\left[-\exp(x\cos t)\sin(nt)\right]_0^\pi + \int_0^\pi n\exp(x\cos t)\cos(nt)\,\mathrm{d}t\right) \\
& && \text{(integration by parts)} \\
&= \frac{2n}{\pi}\int_0^\pi \exp(x\cos t)\cos(nt)\,\mathrm{d}t \\
&= 2nI_n(x)
\end{aligned}$$

□

Corollary A.4. *For all $x \in \mathbb{R}$ and $n \in \mathbb{R}$,* $\quad xI_n'(x) = xI_{n-1}(x) - nI_n(x)$

Proof.

$$\begin{aligned}
xI_n'(x) &= \frac{x}{2}\left(I_{n+1}(x) + I_{n-1}(x)\right) && \text{(Lemma A.2)} \\
&= \frac{1}{2}\left(2xI_{n-1}(x) - 2nI_n(x)\right) && \text{(Lemma A.3)} \\
&= xI_{n-1}(x) - nI_n(x)
\end{aligned}$$

□

Corollary A.5. *For all $x \in \mathbb{R}$ and $n \in \mathbb{R}$,* $\quad xI_n'(x) = xI_{n+1}(x) + nI_n(x)$

Proof.

$$\begin{aligned} xI_n'(x) &= \frac{x}{2}\left(I_{n+1}(x) + I_{n-1}(x)\right) && \text{(Lemma A.2)} \\ &= \frac{1}{2}\left(2xI_{n+1}(x) + 2nI_n(x)\right) && \text{(Lemma A.3)} \\ &= xI_{n+1}(x) + nI_n(x) \end{aligned}$$

□

Proposition A.6. *For all* $x \in \mathbb{R}^*$ *and* $n \in \mathbb{R}$, $\quad x^2 I_n''(x) + xI_n'(x) - (x^2+n^2)I_n(x) = 0$

Proof. Using Corollary A.4, we have $\quad x^2 I_n'(x) = x^2 I_{n-1}(x) - nxI_n(x)$
Deriving this expression with respect to x, we get

$$x^2 I_n''(x) + 2xI_n'(x) = 2xI_{n-1}(x) + x^2 I_{n-1}'(x) - nI_n(x) - nxI_n'(x)$$

$$\begin{aligned} x^2 I_n''(x) + xI_n'(x) &= 2xI_{n-1}(x) + x^2 I_{n-1}'(x) - nI_n(x) - (n+1)xI_n'(x) \\ &= 2xI_{n-1}(x) + x\left(xI_n(x) + (n-1)I_{n-1}(x)\right) - nI_n(x) - (n+1)xI_n'(x) && \text{(Corollary A.5)} \\ &= x^2 I_n(x) + (n+1)xI_{n-1}(x) - nI_n(x) - (n+1)xI_n'(x) \\ &= x^2 I_n(x) + (n+1)xI_{n-1}(x) - nI_n(x) - (n+1)\left(xI_{n-1}(x) - nI_n(x)\right) && \text{(Corollary A.4)} \\ &= x^2 I_n(x) - nI_n(x) + (n+1)nI_n(x) \\ &= (x^2+n^2)I_n(x) \end{aligned}$$

□

Corollary A.7. *Let* $a \in \mathbb{R}$ *and* $J_{n,a}(x) = I_n(ax)$. *Then,*

$$x^2 J_{n,a}''(x) + xJ_{n,a}'(x) - \left((ax)^2 + n^2\right) J_{n,a}(x) = 0$$

Theorem A.8. *For all* $a > 0$ *and all* $K \in \mathbb{N}$,

$$J_{n,a}(x) = \frac{\exp(ax)}{\sqrt{2\pi ax}}\left(\sum_{k=0}^{K-1} \frac{\alpha_{n,k}}{(ax)^k} + \underset{x\to+\infty}{O}\left(\frac{1}{x^K}\right)\right)$$

where
$$\alpha_{n,k} = \frac{(-1)^k \prod_{j=1}^{k}\left(4n^2 - (2j-1)^2\right)}{8^k k!}$$

Proof. We have $\quad J_{n,a}(x) = \frac{1}{\pi}\int_0^{\pi} \exp(ax\cos u)\cos(nu)\,\mathrm{d}u$

Taking $x.0$, with the change of variable $v = \frac{1}{2}axu^2$, we get

$$\begin{aligned} J_{n,a}(x) &= \frac{\exp(ax)}{\pi}\int_0^{\frac{ax\pi^2}{2}} \exp(-v)\exp\left(\sum_{k=2}^{+\infty}\frac{(-2v)^k}{(2k)!(ax)^{k-1}}\right)\cos\left(n\sqrt{\frac{2v}{ax}}\right)\frac{1}{\sqrt{2axv}}\,\mathrm{d}v \\ &= \frac{\exp(ax)}{\pi\sqrt{2ax}}\int_0^{\frac{ax\pi^2}{2}} \frac{\exp(-v)}{\sqrt{v}}\sum_{k=0}^{+\infty}\frac{P_{n,k,a}(v)}{(ax)^k}\,\mathrm{d}v \end{aligned}$$

where $P_{n,k,a}(v)$ is a polynomial (in v) of degree k and such that $P_{n,0,a}(v) = 1$. Therefore,

$$\begin{aligned} J_{n,a}(x) &= \frac{\exp(ax)}{\pi\sqrt{2ax}}\left(\sum_{k=0}^{K-1}\int_0^{\frac{ax\pi^2}{2}} \frac{\exp(-v)}{\sqrt{v}}\frac{P_{n,k,a}(v)}{(ax)^k}\,\mathrm{d}v + \underset{x\to+\infty}{O}\left(\frac{1}{x^K}\right)\right) \\ &= \frac{\exp(ax)}{\pi\sqrt{2ax}}\left(\sum_{k=0}^{K-1}\int_0^{+\infty} \frac{\exp(-v)}{\sqrt{v}}\frac{P_{n,k,a}(v)}{(ax)^k}\,\mathrm{d}v + \underset{x\to+\infty}{O}\left(\frac{1}{x^K}\right)\right) \end{aligned}$$

We recognize the Γ function of half integers with some coefficients. The first one is exactly $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Hence, there are coefficients $\alpha_{n,k,a}$, with $\alpha_{n,0,a} = 1$, such that

$$J_{n,a}(x) = \frac{\exp(ax)}{\sqrt{2\pi ax}}\left(\sum_{k=0}^{K-1}\frac{\alpha_{n,k,a}}{(ax)^k} + \underset{x\to+\infty}{O}\left(\frac{1}{x^K}\right)\right)$$

By the usual properties of asymptotic developments, we can deduce that the formal power series

$$\mathcal{J}_{n,a}(x) := \frac{\exp(ax)}{\sqrt{2\pi ax}} \sum_{k=0}^{+\infty} \frac{\alpha_{n,k,a}}{(ax)^k}$$

must satisfy the same differential equation as $J_{n,a}$, for instance, the one from Corollary A.7. Notice that this series may be nowhere convergent, that is why we insist on the fact that it is a formal power series.

$$\begin{aligned} 0 &= x^2 \mathcal{J}''_{n,a}(x) + x\mathcal{J}'_{n,a}(x) - \left((ax)^2 + n^2\right) \mathcal{J}_{n,a}(x) \\ &= \frac{\exp(ax)}{4\sqrt{2\pi}} \left(\sum_{k=0}^{\infty} \left(4k^2 + 4k + 1\right) \frac{\alpha_{n,k,a}}{(ax)^{k+\frac{1}{2}}} - 4ax \sum_{k=0}^{\infty} (2k+1) \frac{\alpha_{n,k,a}}{(ax)^{k+\frac{1}{2}}} + \left(4ax - 4n^2\right) \sum_{k=0}^{\infty} \frac{\alpha_{n,k,a}}{(ax)^{k+\frac{1}{2}}} \right) \end{aligned}$$

Thus,

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} \left(4k^2 + 4k + 1 - 4n^2\right) \frac{\alpha_{n,k,a}}{(ax)^{k+\frac{1}{2}}} - \sum_{k=-1}^{\infty} 8(k+1) \frac{\alpha_{n,k+1,a}}{(ax)^{k+\frac{1}{2}}} \\ &= \sum_{k=0}^{\infty} \left((2k+1)^2 - 4n^2\right) \frac{\alpha_{n,k,a}}{(ax)^{k+\frac{1}{2}}} - \sum_{k=0}^{\infty} 8(k+1) \frac{\alpha_{n,k+1,a}}{(ax)^{k+\frac{1}{2}}} \end{aligned}$$

Since we are studying a formal power series, this equality means that all the coefficients must cancel out. In other words, for all $k \in \mathbb{N}$,

$$-\left(4n^2 - (2k+1)^2\right) \alpha_{n,k,a} = 8(k+1)\alpha_{n,k+1,a}$$

Note that $\alpha_{n,0,a} = 1$ and does not depend on a. Therefore, for any natural number k, $\alpha_{n,k,a}$ does not depend on a neither. We then drop out the index a. By immediate induction, we get

$$\alpha_{n,k} = \frac{(-1)^k \prod_{j=1}^{k} \left(4n^2 - (2j-1)^2\right)}{8^k k!}$$

what concludes the proof.

□

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Index

ECOLE DOCTORALE

Titre : Calculs avec la droite réelle généralisée

Mots clés : Calcul analogique, Nombres Surréels, Corps réels clos, Calculs sur les réels

Résumé : Les nombres surréels sont une classe de nombres très particulière dans laquelle il est possible d'injecter absolument tous les corps ordonnés. En particulier, les transseries, étudiées notamment pour le travail sur les asymptotiques de fonctions réelles, sont un exemple d'un tel corps. Les nombres surréels contiennent cependant encore bien plus de nombres. Typiquement, ils contiennent des nombres qui peuvent représenter des fonctions hyper exponentielles ou au contraire demi-exponentielles (c'est-à-dire dont la composée par elle même est la fonction exponentielle). Par ailleurs, les nombres surréels peuvent être dérivés et primitivés. Ainsi, il est possible de former des équations différentielles dont les solutions sont des nombres surréels. De telles solutions à des équations différentielles peuvent alors être vues comme des asymptotiques de systèmes différentiels.
D'autre part, il a été prouvé qu'il est possible de simuler les machines de Turing grâce aux équations différentielles polynomiales à plusieurs dimensions et que celles-ci modélisent l'évolution d'un ordinateur analogique (ou GPAC, pour General Purpose Analog Computer). Dans un tel modèle, la condition initial est donnée comme entrée du système et la sortie est donnée par la limite d'une ou plusieurs coordonnées.
L'objectif initiale de cette thèse était de mettre en lien ces deux mondes et d'étudier comment les nombres surréels peuvent être utiles du point de vue de la théorie de la calculabilité. Notre travail nous amenés à considérer des corps de nombres surréels par nature beaucoup plus petits que la classe des nombres surréels elle-même. En effet, en prenant des corps de plus en plus gros, chaque corps peut apporter des informations sur le précédent et il est possible d'utiliser des nombres infiniment grands ou infiniment petits par rapport au corps étudié. Par définition, ceci est impossible si nous considérons la classe des nombres surréels en entier.
Dans cette thèse, nous avons donc identifié des corps de nombres surréels stables par les opérations d'exponentiation, de logarithme, de dérivation, et de primitivation. De plus, nous avons pu construire des corps qui n'utilisent pas de nombres excessivement grands comme, par exemple, les ordinaux indénombrables ou incalculables. Les corps que nous avons construits sont définis par des séries formelles et non par la longueur des nombres surréels. En effet, la connaissance de la forme normale des nombres surréels sous forme de série formelle apporte, selon nous, beaucoup plus d'information sur les fonctions que les nombres surréels sont sensés représenter.
Ayant identifié des tels corps, nous proposons aussi une piste de travail pour étudier les comportement oscillants des systèmes différentiels et introduisons ainsi un nouveau type de nombres, construit sur les corps de nombres surréels que nous avons construits, les nombres oscillants.

Title : Computations with the generalized real line

Keywords : Analog Computation, Surreal numbers, Real-closed fields, Computations over the reals

Abstract : Surreal numbers form a very singular class of number in which we can embed every ordered field. In particular, transseries, studied for instance in Asymptotic Theory, form examples of such fields. Surreal numbers are in fact an even larger field which contains numbers that represent hyper exponential functions or half exponential functiosn (whose composition by themselves is the exponential function). It is also possible to get the derivative and the anti-derivative of surreal numbers. Thus, we can study differential equations whose solutions are surreal numbers. We then can see such solutions as the asymptotics to some differential systems.
From another side, it has been proved that Turing machines can be emulated with multidimensional polynomial differential equations and that these later model caracterizes the evolution of analog computers (or GPAC for General Purpose Analog Computer). In such model, the initial condition of system is seen as the input and the output is given by the limits of one or several components of the system.
The initial motivation of the thesis was to draw the link between these points of view and study how surreal numbers can be useful to Computability Theory. This work lead us to consider fields of surreal numbers much smaller that the whole class of surreal numbers. Doing that and considering larger and larger fields, we can get more and more information on the fields thanks to infinitely large of infinitely small numbers with respect to the studied field. This is, by definition, impossible in the whole class of surreal numbers.
In this thesis, we identified some fields of surreal numbers stable under some useful operations such as the exponentiation, the logarithm, the derivation and the anti-derivation. Moreover, we have been able to provide a field that does not contain excessively large numbers such as uncountable or uncomputable ordinals. The fields we provide are defined by formal power series instead of the length of surreal numbers. Indeed, the knowledge of the normal form of the surreal numbers provides much more information about the functions the surreal numbers represent.
Such fields being identified, we suggest some ideas to study the oscillating behavior of differential systems and introduce a new kind of numbers, oscillating numbers, built on the surreal fields we provided, to embedded this information.

Institut Polytechnique de Paris
91120 Palaiseau, France